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THERMO-STRUCTURAL ANALYSIS MANUAL

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Aeronautical Systems Division
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Authors: M. J. Forray, M. Newman, and H. Switzky.)

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FOREWORD

This Manual was prepared by the Structural Research and Development Group, Structures Section, Research and Development Division of Republic Aviation Corporation. The work was initiated under Contract AF33(616)-6066 in the 750A Applied Research Program, the Mechanics of Flight, Project No. 1367, Structural Design Criteria, and Task No. 14002, "Structural Analysis Methods". The work is now documented under Task 136710. This work was initiated under the direction of the Structural Analysis Unit, Structures Branch, Aircraft Laboratory, Directorate of Laboratories, Wright Air Development Center.* Mr. I. Winnegrad acted initially as project engineer and was succeeded by Mr. C. Richard. The Manual was completed under the direction of the Structural Analysis Unit, Configuration Research Section, Structures Branch, Flight Dynamics Laboratory, Deputy Commander/Technology, Aeronautical Systems Division, with Mr. G. E. Maddux as Project Engineer.

The work was coordinated and supervised by Dr. R. S. Levy, Head of the Structural Research and Development Group. His valuable suggestions and criticisms are gratefully acknowledged as are those of the following personnel of the Applied Research and Development Division of Republic Aviation Corporation: Mr. A. Alberi, Acting Manager of Technical Engineering; Mr. C. Rosenkranz, Acting Chief Structures Engineer; and Mr. C. Meissner, Principal Structures Engineer.

NOTES ON USING THE MANUAL

This Manual consists of five basic sections, divided into numbered sub-sections and paragraphs. For simplicity in cross-referencing material in the text, all portions of the Manual designated with a two-tier number (e.g., 1.1) are considered sub-sections, and all portions designated by numbers of three or more tiers (e.g., 1.1.1 or 1.1.1.1) are considered paragraphs.

Throughout the Manual, the numbered paragraphs (or sub-sections) have been used as the basis for numbering figures, tables, and equations, with new sequences beginning with each numbered paragraph. Figure and table numbers consist of an appropriate paragraph number, followed by a sequence number for the particular figure or table. For convenience the paragraph designations have been omitted from the equation numbers. When an equation from another paragraph is cited in the text, the number of the paragraph in which that equation occurs is also cited. When a paragraph number is not given in conjunction with the citation of an equation, it is to be assumed that the equation is included in the paragraph in which the citation occurs.

References are listed at the end of those sections which have more than one reference. In addition, each section contains its own complete table of contents and list of symbols.

*Now under direction of Flight Dynamics Laboratory, Directorate of Aeromechanics, Aeronautical Systems Division.

ABSTRACT

This second volume of the Thermo-Structural Analysis Manual considers additional problems in the field of thermal and mechanical stress analysis not fully treated in Volume I. Special emphasis is given to nonlinear analysis of beams and plates and to axisymmetric thermo-elastic analysis of thin shells. Following the format of Volume I, nondimensional graphs, formulas and tables are developed where feasible. For clarification of the analytical techniques and the use of the numerical data, illustrative examples are given.

The following problems are treated in five individual sections:

- (1) [Large deflection analysis of straight elastic beams with axial end restraint and axial end loads coupled with transverse loading and temperature,]
- (2) [Approximate determination of the axial end loads and deformations in heated beam columns with initial eccentricities,]
- (3) [Approximate solutions for the buckling of eccentric columns accommodating nonlinear stress-strain laws,]
- (4) [Axisymmetric large deflections of circular plates subjected to thermal and mechanical loads,]
- (5) [Axisymmetric thermo-elastic analysis of thin shells.]

PUBLICATION REVIEW

This publication has been reviewed and is approved.

FOR THE COMMANDER:



R. F. HOXNER
Chief, Structures Branch
Flight Dynamics Laboratory

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INTRODUCTION

This Volume II of the Manual is an extension of the basic work presented in Volume I and covers the following special problem areas in heated beams, plates and shells:

- (1) Nonlinear beam analyses including the effects of initial eccentricities and nonlinear material properties.
- (2) Elastic circular plates loaded and heated axisymmetrically, considering the effects of large deflections
- (3) The axisymmetric thermoelastic analysis of shells developed in Volume I is generalized, removing the restrictions on geometry and including the effects of arbitrary temperature variations through the thickness.

The material is presented in the form of five independent reports or sections.

The emphasis has been placed on the development of analytical techniques and formulas with their corresponding physical interpretations. Where feasible, the investigations are self-contained, starting with fundamental theoretical considerations and notions in the field of static thermo-structural analysis. However, the reader may find it useful to refer to Volume I where a systematic development of the basic concepts is given.

Brief summaries of the five sections of this volume of the Manual follow.

Section 1. Beam Columns Subjected to Elevated Temperature and Mechanical Loads

This section treats beam columns with axial end restraints and loads coupled with transverse loads and temperature gradients. Numerical results, "exact" within the framework of large deflection beam theory are tabulated for certain cases.

Section 2. Approximate Solution for an Axially Restrained Column Subjected to Elevated Temperature and Lateral Load

This section presents an approximate method of solution of the heated beam column problem. The method is amenable to problems involving spanwise variations of load thermal gradients and stiffness and permits treatment of initial eccentricities.

Section 3. Approximate Solution for the Buckling of Eccentric Columns

This section develops and presents nondimensional curves to predict column buckling loads. Nonlinear stress-strain relationships are accommodated as well as lateral loads and initial eccentricities or thermally induced deformations.

Section 4. Axisymmetric Large Deflections of Circular Plates Subjected to Thermal and Mechanical Loads

The interaction of membrane stresses and bending is considered for axially restrained circular plates subjected to heat and load. The differential equations governing the axisymmetric case are derived and an iterative digital scheme is used to obtain numerical results for a special case over a wide range of temperature and load parameters.

Section 5. Axisymmetric Stresses and Deflections in Shells Due to Thermal and Mechanical Loads

The general equation for linear elastic analysis of axisymmetric shell problems is developed including the effects of temperature and load. The special cases of conical and cylindrical shells are developed and a numerical example is given for a cylinder with temperature gradients through the thickness and along the length.

SECTION 1
BEAM COLUMNS SUBJECTED TO ELEVATED TEMPERATURE
AND MECHANICAL LOADS

by

M. Forray
M. Newman

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1.0

SECTION 1
BEAM COLUMNS SUBJECTED TO ELEVATED TEMPERATURE
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SECTION 1

BEAM COLUMNS SUBJECTED TO ELEVATED TEMPERATURE

AND MECHANICAL LOADS

1.1. SUMMARY

This report considers the nonlinear analysis (large deflections) of beams with axial end restraints and axial end loads coupled with transverse loading and temperature.

In order to keep the number of parameters within practical limits the following conditions are investigated:

- (1) Distributed transverse loads are uniform over the span while concentrated loads are at the midspan.
- (2) The temperature varies linearly through the depth and is constant in a span-wise direction.
- (3) The beam is assumed to be simply supported at its ends for bending and elastically restrained axially (Figure 1.3-1).

Tables of numerical results in nondimensional form are presented for the cases of zero and full axial end restraint in rectangular beams. These tables may be used to determine maximum deflections and bending moments.

1.2 INTRODUCTION

In structural analysis beam columns differ from simple beams in that the addition of axial end loads and restraints interact with transverse loads in a nonlinear manner, thus invalidating the principle of superposition (Reference 1-1).

When the axial end loads are specified and the beam is unrestrained axially, the solution for bending moments and deflections are obtained, in general, by solving a linear, non-homogeneous differential equation with constant coefficients subject to appropriate boundary conditions. If, in addition, the beam is restrained axially, the total end load is an unknown and an additional compatibility relation must be employed.

The analysis presented considers a beam of constant cross section for which the Bernoulli-Euler assumption of classical beam theory is employed. This implies that plane sections perpendicular to the centroidal axis before bending remain plane and perpendicular to the deflected centroidal axis. It is further assumed that the material behavior is linearly elastic.

It can be shown that the seemingly approximate technique used below (Sub-section 1.3) yields results which are identical to those obtained by a moderately large deflection analysis in which nonlinear strain-displacement relations are employed.

1.2 (Cont'd)

The following symbols are used throughout this section:

b	Width of rectangular beam
h	Depth of beam
x	Spanwise coordinate
y	Vertical deflection
\bar{y}	Nondimensional central deflection
A	Cross sectional area
E	Young's modulus
H	Magnitude of axial load in beam
I	Moment of inertia
$2K$	Spring stiffness of axial end restraint
L	Half beam length
M	Bending moment
\bar{M}	Nondimensional central bending moment
P	Known applied axial end load
$2Q$	Concentrated midspan transverse load
\bar{Q}, \bar{Q}	Nondimensional concentrated midspan load
T_i, T_o	Temperatures at lower and upper beam faces, respectively
T_d, \bar{T}_d	Nondimensional temperature differences between upper and lower beam faces
\bar{T}	Nondimensional average temperature
\bar{W}	Intensity of uniformly distributed transverse load
\bar{W}, \bar{W}	Nondimensional uniformly distributed load
α	Coefficient of linear thermal expansion
β	Ratio of distance from lower beam face to centroidal axis to the total depth
λ	$\pm \sqrt{\frac{H}{EI}}$
$\bar{\lambda}$	λL , nondimensional
σ	stress

SUBSCRIPTS

M	Due to mechanical loads
T	Due to temperature

1.3 DERIVATION OF BASIC EQUATIONS

The beam (Figure 1.3-1) is referred to a rectangular coordinate system in which transverse deflections y and distances along the centroidal axis are measured from a point on the undeflected centroidal axis at mid-span. Specified axial end loads are denoted by P and the elastic axial restraints are shown schematically as springs with stiffness $2K$. We assume uniform distributed loads of intensity W and a mid-span concentrated load of magnitude $2Q$. The temperature variation through the thickness is linear, varying from T_i at the lower extreme fiber to T_o at the upper extreme fiber.

Since the temperature distribution varies linearly with respect to Cartesian coordinates it produces no stresses in an externally unrestrained beam (Reference 1-2). In this case, the stress-free thermal curvature is given by

$$y_T''' = -\frac{\alpha(T_o - T_i)}{h} \quad (1)$$

1.3 (Cont'd)

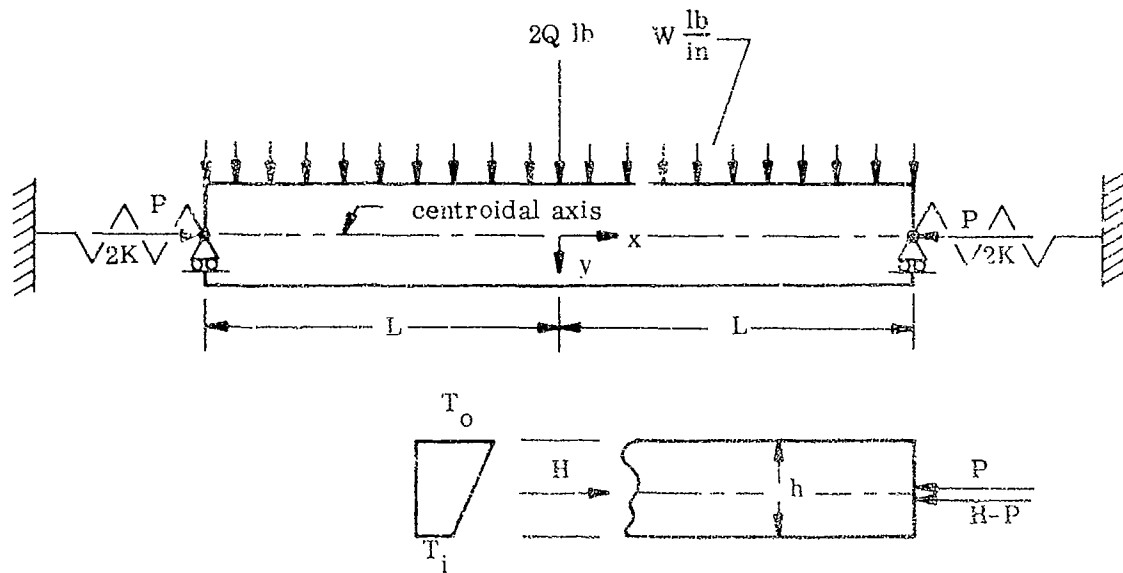


FIGURE 1.3-1 BEAM COLUMN - MODEL USED FOR ANALYSIS

We now consider that the external specified and redundant mechanical loads are applied to this thermally bent beam. Designating the additional curvatures produced by these mechanical loads as y''_M , it follows that

$$y''_M = \frac{-M}{EI} = \frac{1}{EI} \left[\pm Hy - (WL + Q)(L - x) + \frac{W}{2}(L - x)^2 \right], \quad (2)$$

where positive moments cause compression in the outer fibers, y is the total deflection, and H is the magnitude of the axial load in beam*. The negative and positive signs in the first term of the right hand side of Eq. (2) refer to compression and tension respectively. Adding Eqs. (1) and (2) and rearranging yields

$$y'' \pm \frac{Hy}{EI} = \frac{W}{2EI}(L - x)^2 - \frac{(WL + Q)(L - x)}{EI} + \frac{\alpha}{h}(T_o - T_i). \quad (3)$$

Due to symmetry it is only necessary to consider one half of the beam, so that the differential equation given by Eq. (3) applies in the interval $0 \leq x \leq L$ and $y(x)$ is an even function, i.e., $y(x) = y(-x)$. The boundary conditions are given by

$$y'(0) = y(L) = 0. \quad (4)$$

1.4 SOLUTION OF EQUATIONS

The complete solutions of Eqs. (3) and (4) of Sub-section 1.3 for the cases of axial compression and tension, respectively, are written as follows.

* H is always taken as positive, regardless of whether the axial load is tensile or compressive.

1.4 (Cont'd)

(1) Axial Compression

$$y = \frac{Q}{EI\lambda^3} \frac{\sin \lambda (L-x)}{\cos \lambda L} + \frac{\cos \lambda x}{\lambda^2 \cos \lambda L} \left[\frac{W}{EI\lambda^2} - \frac{\alpha}{h} (T_o - T_i) \right] + \frac{Wx^2}{2EI\lambda^2} + \frac{Qx}{EI\lambda^2} + \frac{1}{\lambda^2} \left[\frac{\alpha}{h} (T_o - T_i) - \frac{QL}{EI} - \frac{WL^2}{2EI} - \frac{W}{EI\lambda^2} \right] \quad (1a)$$

(2) Axial Tension

$$y = \frac{Q}{EI\lambda^3} \frac{\sinh \lambda (x-L)}{\cosh \lambda L} + \frac{\cosh \lambda x}{\lambda^2 \cosh \lambda L} \left[\frac{W}{EI\lambda^2} + \frac{\alpha}{h} (T_o - T_i) \right] - \frac{Wx^2}{2EI\lambda^2} - \frac{Qx}{EI\lambda^2} - \frac{1}{\lambda^2} \left[\frac{\alpha}{h} (T_o - T_i) - \frac{QL}{EI} - \frac{WL^2}{2EI} + \frac{W}{EI\lambda^2} \right] \quad (1b)$$

where

$$\lambda = -\sqrt{\frac{H}{EI}} \quad \text{for compression,} \quad (1c)$$

$$\lambda = +\sqrt{\frac{H}{EI}} \quad \text{for tension.}$$

The solution is not yet complete since, in general the axial end load H and hence λ is unknown. An additional compatibility relationship must be employed to evaluate this quantity. Such a relationship is obtained by noting that the change in the spanwise distance between the beam ends due to bending, thermal expansion and mechanical axial strains must be equal to the total change in length of the axial end restraints. This condition can be expressed by

$$\frac{1}{2} \int_{-L}^L (y')^2 dx + \frac{HL}{AE} - \alpha L \left\{ \beta T_o + (1 - \beta) T_i \right\} + \frac{(HL - P)L}{2K} = 0 \quad (3)$$

where positive and negative signs associated with H refer to compression and tension respectively and β defines the location of the centroidal axis (Figure 1.4-1).

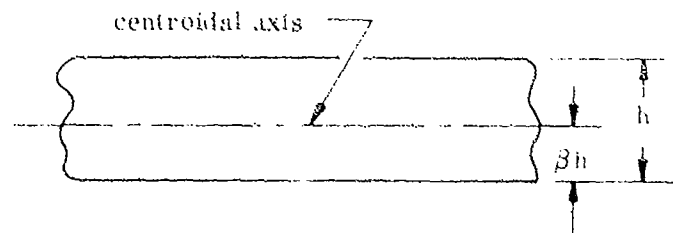


FIGURE 1.4-1 PARAMETER β , DEFINING CENTROIDAL AXIS LOCATION

1.4 (Cont'd)

Substitution of Eqs. (1a) and (1b) into (3) and simplifying yields for the compression case

$$\begin{aligned}
 & \frac{\bar{Q}^2}{\bar{\lambda}^5} \left[(\sin \bar{\lambda}) \left(\cos^2 \frac{\bar{\lambda}}{2} - \frac{5}{2} \right) + \frac{3\bar{\lambda}}{2} \right] \\
 & + \left[\frac{1}{2} - \frac{\sin 2\bar{\lambda}}{4\bar{\lambda}} \right] \left[\frac{1}{\cos^2 \bar{\lambda}} \right] \left[\frac{\bar{W}}{\bar{\lambda}^3} + \frac{\bar{Q} \sin \bar{\lambda}}{\bar{\lambda}^2} - \frac{\bar{T}_d}{\bar{\lambda}} \right]^2 \\
 & + \frac{\bar{W}^2}{3\bar{\lambda}^4} - \left[\frac{\bar{W}}{\bar{\lambda}^3} + \frac{\bar{Q} \sin \bar{\lambda}}{\bar{\lambda}^2} - \frac{\bar{T}_d}{\bar{\lambda}} \right] \left[\frac{4\bar{Q}}{3\bar{\lambda} \cos \bar{\lambda}} \sin^4 \left(\frac{\bar{\lambda}}{2} \right) \right] \\
 & + \frac{\bar{Q}\bar{W}}{\bar{\lambda}^4} \left[1 - 2 \left(\frac{\sin \bar{\lambda}}{\bar{\lambda}} + \frac{\cos \bar{\lambda}}{\bar{\lambda}^2} - \frac{1}{\bar{\lambda}^2} \right) \right] \\
 & - \frac{2\bar{W}}{\bar{\lambda}^4 \cos \bar{\lambda}} (\sin \bar{\lambda} - \bar{\lambda} \cos \bar{\lambda}) \left[\frac{\bar{W}}{\bar{\lambda}^3} + \frac{\bar{Q} \sin \bar{\lambda}}{\bar{\lambda}^2} - \frac{\bar{T}_d}{\bar{\lambda}} \right] \\
 & + \left[\frac{2I}{AL^2} + \frac{EI}{KL^3} \right] \bar{\lambda}^2 = \frac{P}{KL} + 2\alpha \left[\beta T_o + (1 - \beta) T_i \right] \quad , \quad (4a)
 \end{aligned}$$

and for the case of axial tension:

$$\begin{aligned}
 & \frac{\bar{Q}^2}{\bar{\lambda}^5} \left[(\sinh \bar{\lambda}) \left(\cosh^2 \frac{\bar{\lambda}}{2} - \frac{5}{2} \right) + \frac{3\bar{\lambda}}{2} \right] \\
 & - \left[\frac{1}{2} - \frac{\sinh 2\bar{\lambda}}{4\bar{\lambda}} \right] \left[\frac{1}{\cosh^2 \bar{\lambda}} \right] \left[\frac{\bar{W}}{\bar{\lambda}^3} - \frac{\bar{Q} \sinh \bar{\lambda}}{\bar{\lambda}^2} + \frac{\bar{T}_d}{\bar{\lambda}} \right]^2 \\
 & + \frac{\bar{W}^2}{3\bar{\lambda}^4} + \left[\frac{4\bar{Q} \sinh^4 \frac{\bar{\lambda}}{2}}{\bar{\lambda}^3 \cosh \bar{\lambda}} \right] \left[\frac{\bar{W}}{\bar{\lambda}^3} - \frac{\bar{Q} \sinh \bar{\lambda}}{\bar{\lambda}^2} + \frac{\bar{T}_d}{\bar{\lambda}} \right] \\
 & + \frac{\bar{Q}\bar{W}}{\bar{\lambda}^4} \left[1 - 2 \left(\frac{\sinh \bar{\lambda}}{\bar{\lambda}} - \frac{\cosh \bar{\lambda}}{\bar{\lambda}^2} + \frac{1}{\bar{\lambda}^2} \right) \right] \\
 & + \frac{2\bar{W}}{\bar{\lambda}^4 \cosh \bar{\lambda}} (\sinh \bar{\lambda} - \bar{\lambda} \cosh \bar{\lambda}) \left[\frac{\bar{W}}{\bar{\lambda}^3} - \frac{\bar{Q} \sinh \bar{\lambda}}{\bar{\lambda}^2} + \frac{\bar{T}_d}{\bar{\lambda}} \right] \\
 & - \left(\frac{2I}{AL^2} + \frac{EI}{KL^3} \right) \bar{\lambda}^2 = \frac{P}{KL} + 2\alpha \left[\beta T_o + (1 - \beta) T_i \right] \quad , \quad (4b)
 \end{aligned}$$

1.4 (Cont'd)

where

$$\frac{QL^2}{EI} = \tilde{Q}$$

$$\frac{WL^3}{EI} = \tilde{W}$$

$$\begin{aligned} \lambda L = \bar{\lambda} &= + \sqrt{\frac{HL^2}{EI}} \quad \text{for tension} \\ &= - \sqrt{\frac{HL^2}{EI}} \quad \text{for compression} \end{aligned}$$

$$\frac{\alpha L (T_o - T_i)}{h} = \tilde{T}_d$$

Equations (4a) and (4b) are transcendental equations from which $\bar{\lambda}$ may be determined. Equations (1a) and (1b) then yield the deflections for the known values of $\bar{\lambda}$. In general, Eqs. (4) are difficult to solve except by graphical or trial and error procedures. In addition, presentation of numerical results for the large number of parameters appearing in this general problem is cumbersome. For these reasons, the number of parameters to be used in the presentation of numerical results have been reduced by restricting considerations to the following:

- (1) The beam cross section is rectangular and therefore $\beta = \frac{1}{2}$.
- (2) The beam is prevented from moving axially at its ends ($K \rightarrow \infty$) or free to move axially ($K = 0$).
- (3) The transverse load is either uniform over the span or concentrated at the mid-span.

1.5 NUMERICAL RESULTS FOR BEAMS OF RECTANGULAR CROSS SECTION

CASE A - Uniform transverse load over the span with ends rigidly restrained axially:

The beam is shown schematically in Figure 1.5-1. For this case, we substitute $\tilde{Q} = 0$, $\beta = \frac{1}{2}$, and $\frac{1}{K} = 0$ into Eqs. (1) and (4) of Sub-section 1.4. Defining nondimensional quantities:

$$\begin{aligned} \bar{W} &= \frac{12W}{Eb} \left(\frac{L}{h} \right)^4 \\ \bar{\lambda} &= + \sqrt{\frac{HL^2}{EI}} \quad (\text{tension positive}) \end{aligned}$$

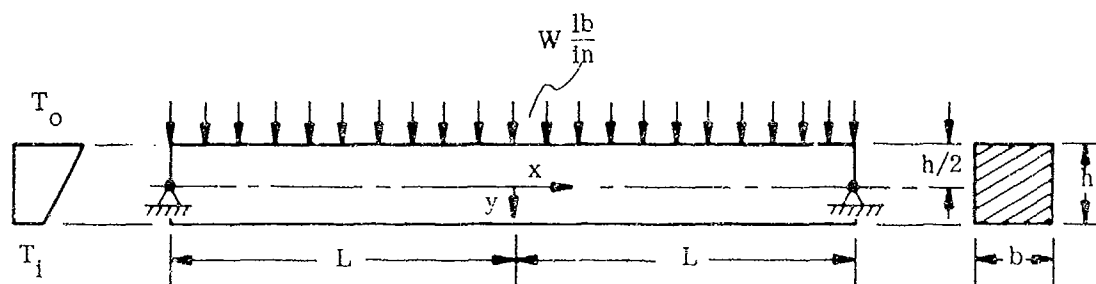


FIGURE 1.5-1 RECTANGULAR BEAM WITH UNIFORM LOAD OVER THE SPAN AND ENDS RIGIDLY RESTRAINED AXIALLY

$$\begin{aligned}
 \bar{T}_d &= \alpha \left(\frac{L}{h} \right)^2 (T_0 - T_1) \\
 \bar{T} &= \alpha \left(\frac{L}{h} \right)^2 (T_0 + T_1) \\
 \bar{y} &= \left[\frac{y}{h} \right]_{x=0} \\
 \bar{M} &= \left[\frac{12ML^2}{Ebh^4} \right]_{x=0}
 \end{aligned} \tag{1}$$

These equations yield

$$\left[\frac{1}{2} - \frac{\sin 2\bar{\lambda}}{4\bar{\lambda}} \right] \left[\frac{1}{\bar{\lambda}^2 \cos 2\bar{\lambda}} \right] \left[\frac{\bar{W}}{\bar{\lambda}^2} - \bar{T}_d \right]^2 + \frac{\bar{W}^2}{3\bar{\lambda}^4} - \frac{2\bar{W}}{\bar{\lambda}^5 \cos \bar{\lambda}} (\sin \bar{\lambda} - \bar{\lambda} \cos \bar{\lambda}) \left[\frac{\bar{W}}{\bar{\lambda}^2} - \bar{T}_d \right] + \frac{\bar{\lambda}^2}{6} = \bar{T} \tag{2a}$$

$$\begin{aligned}
 \bar{y} &= \frac{1}{\bar{\lambda}^2} \left[\frac{\bar{W}}{\bar{\lambda}^2} - \bar{T}_d \right] \left[\frac{1}{\cos \bar{\lambda}} - 1 \right] - \frac{\bar{W}}{2\bar{\lambda}^2} \\
 \bar{M} &= \left[\frac{\bar{W}}{\bar{\lambda}^2} - \bar{T}_d \right] \left[\frac{1}{\cos \bar{\lambda}} - 1 \right]
 \end{aligned} \tag{3a}$$

1.5 (Cont'd)

for the case of compressive end loads, while for the tensile case

$$\begin{aligned}
 & - \left[\frac{1}{2} - \frac{\sinh 2\bar{\lambda}}{4\bar{\lambda}} \right] \left[\frac{1}{\bar{\lambda}^2 \cosh^2 \bar{\lambda}} \right] \left[\frac{\bar{W}}{\bar{\lambda}^2} + \bar{T}_d \right]^2 + \frac{\bar{W}^2}{3\bar{\lambda}^4} \\
 & + \frac{2\bar{W}}{\bar{\lambda}^5 \cosh \bar{\lambda}} (\sinh \bar{\lambda} - \bar{\lambda} \cosh \bar{\lambda}) \left[\frac{\bar{W}}{\bar{\lambda}^2} + \bar{T}_d \right] - \frac{\bar{\lambda}^2}{6} = \bar{T}
 \end{aligned}
 \tag{2b}$$

$$\begin{aligned}
 \bar{y} &= \frac{1}{\bar{\lambda}^2} \left[\frac{\bar{W}}{\bar{\lambda}^2} + \bar{T}_d \right] \left[\frac{1}{\cosh \bar{\lambda}} - 1 \right] + \frac{\bar{W}}{2\bar{\lambda}^2} \\
 \bar{M} &= - \left[\frac{\bar{W}}{\bar{\lambda}^2} + \bar{T}_d \right] \left[\frac{1}{\cosh \bar{\lambda}} - 1 \right] .
 \end{aligned}
 \tag{3b}$$

Values of $\bar{\lambda}$, \bar{y} , and \bar{M} for various combinations of \bar{T}_d , \bar{T} and \bar{W} are given in Table 1.5-1.

This table permits the determination, either directly or by interpolation, of the axial end loads, maximum (central) deflection and maximum bending moment corresponding to specified temperatures and transverse loading. A typical case, extracted from the table is plotted in Figure 1.5-2, showing the variations in end load, central bending moment and deflection with average temperature when the temperature difference and transverse load are held constant.

The figure shows that extremely large values of \bar{T} are required to raise the compressive end load to values in the neighborhood of the critical Euler value, $\bar{\lambda}_{CR} = -\frac{\pi}{2} \approx -1.57$. This is due to the fact that additional beam expansions caused by increasing the average temperature are accommodated by further bending with very little change of compressive end load.

Thus, although the theoretical end load in a perfectly straight axially restrained column reaches the Euler buckling load when $\bar{T} = \frac{\pi^2}{24} \approx .41$, values of \bar{T} several orders of magnitude higher than this can actually be achieved without excessive stresses or deflections occurring. In this respect thermal buckling differs significantly from buckling caused by deadweight loads, since stresses and deflections tend to become excessively large when mechanical loads exceed the Euler buckling value.

1.5 (Cont'd)

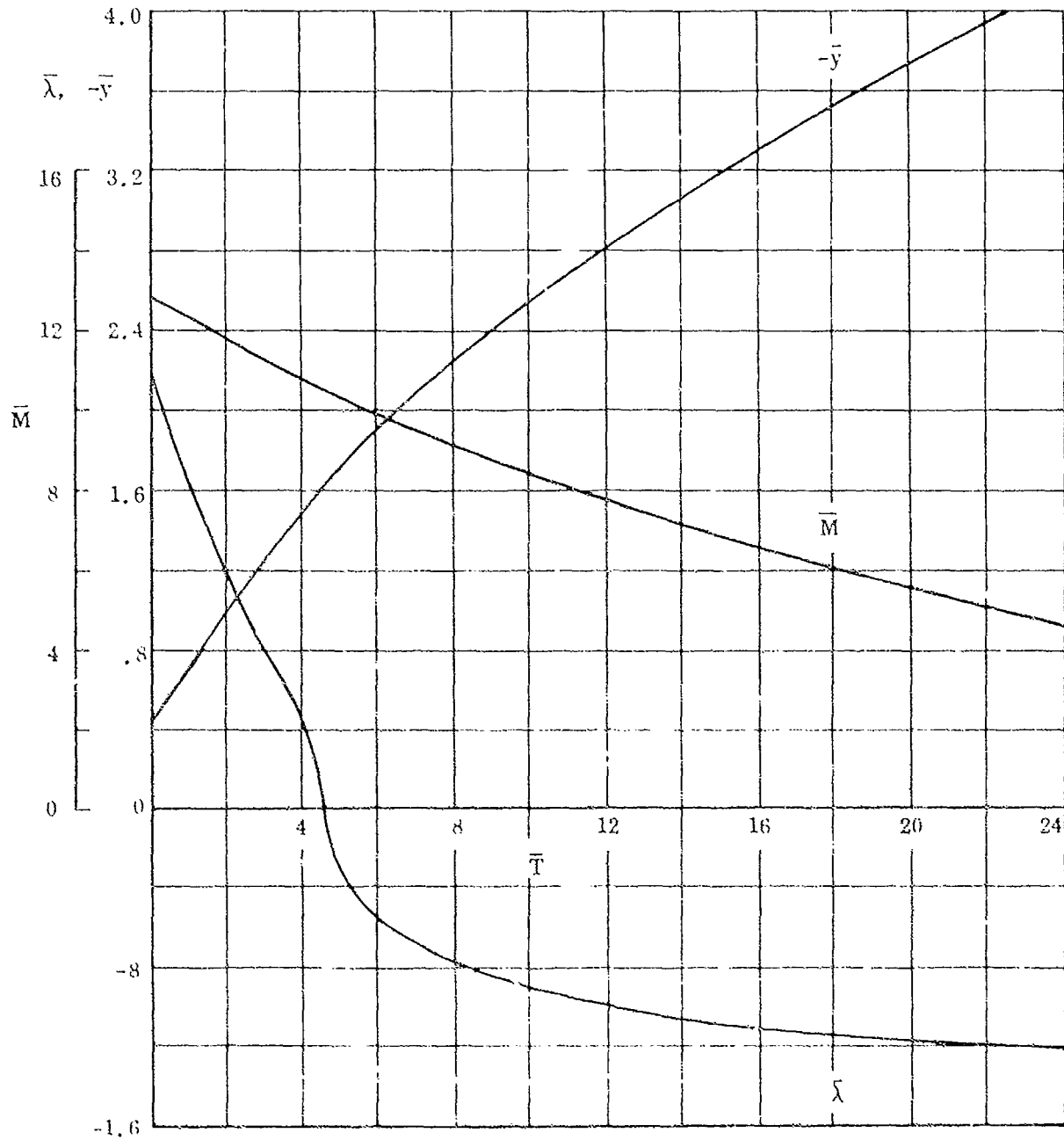


FIGURE 1.5-2 NONDIMENSIONAL END LOAD, CENTRAL DEFLECTION AND MOMENT VS. NONDIMENSIONAL AVERAGE TEMPERATURE; $\bar{W}=21$; $\bar{T}_d=12$

1.5 (Cont'd)

CASE B - Concentrated transverse load at midspan with ends rigidly restrained axially

For this case, we substitute $\bar{W} = 0$, $\beta = \frac{1}{2}$ and $\frac{1}{K} = 0$ into Eqs. (1) and (4) of Sub-section 1.4. Nondimensional quantities are as defined by Eq. (1) and in addition we define

$$\bar{Q} = \frac{12QL^3}{Ebh^4} \quad (4)$$

$$\begin{aligned} \frac{\bar{Q}^2}{\bar{\lambda}^5} & \left[(\sin \bar{\lambda}) \left(\cos^2 \frac{\bar{\lambda}}{2} - \frac{5}{2} \right) + \frac{3\bar{\lambda}}{2} \right] \\ & + \left[\frac{1}{2} - \frac{\sin 2\bar{\lambda}}{4\bar{\lambda}} \right] \left[\frac{1}{\bar{\lambda}^2 \cos^2 \bar{\lambda}} \right] \left[\frac{Q \sin \bar{\lambda}}{\bar{\lambda}} - \bar{T}_d \right]^2 \\ & - \frac{4\bar{Q} \sin^4 \frac{\bar{\lambda}}{2}}{\bar{\lambda}^4 \cos \bar{\lambda}} \left[\frac{Q \sin \bar{\lambda}}{\bar{\lambda}} - \bar{T}_d \right] + \frac{\bar{\lambda}^2}{6} = \bar{T} \end{aligned} \quad (5a)$$

$$\begin{aligned} \bar{y} &= \frac{\bar{Q}}{\bar{\lambda}^2} \left(\frac{\tan \bar{\lambda}}{\bar{\lambda}} - 1 \right) - \frac{\bar{T}_d}{\bar{\lambda}^2} \left(\frac{1}{\cos \bar{\lambda}} - 1 \right) \\ \bar{M} &= \bar{Q} \frac{\tan \bar{\lambda}}{\bar{\lambda}} - \bar{T}_d \left[\frac{1}{\cos \bar{\lambda}} - 1 \right], \end{aligned} \quad (6a)$$

for the case of compressive end loads, while for the tensile case

$$\begin{aligned} \frac{\bar{Q}^2}{\bar{\lambda}^5} & \left[(\sinh \bar{\lambda}) \left(\cosh^2 \frac{\bar{\lambda}}{2} - \frac{5}{2} \right) + \frac{3\bar{\lambda}}{2} \right] \\ & \left[\frac{1}{2} - \frac{\sinh 2\bar{\lambda}}{4\bar{\lambda}} \right] \left[\frac{1}{\bar{\lambda}^2 \cosh^2 \bar{\lambda}} \right] \left[\frac{\bar{Q} \sinh \bar{\lambda}}{\bar{\lambda}} - \bar{T}_d \right]^2 \\ & - \frac{4\bar{Q} \sinh^4 \frac{\bar{\lambda}}{2}}{\bar{\lambda}^4 \cosh \bar{\lambda}} \left[\frac{\bar{Q} \sinh \bar{\lambda}}{\bar{\lambda}} - \bar{T}_d \right] - \frac{\bar{\lambda}^2}{6} = \bar{T} \\ \bar{y} &= - \frac{\bar{Q}}{\bar{\lambda}^2} \left(\frac{\tanh \bar{\lambda}}{\bar{\lambda}} - 1 \right) + \frac{\bar{T}_d}{\bar{\lambda}^2} \left(\frac{1}{\cosh \bar{\lambda}} - 1 \right) \\ \bar{M} &= \bar{Q} \frac{\tanh \bar{\lambda}}{\bar{\lambda}} - \bar{T}_d \left[\frac{1}{\cosh \bar{\lambda}} - 1 \right]. \end{aligned} \quad (6b)$$

Values of $\bar{\lambda}$, \bar{y} , and \bar{M} for various combinations of \bar{T}_d , \bar{T} and \bar{Q} are given in Table 1.5-2.

TABLE 1.5-1
VALUES OF $\bar{\lambda}$, γ AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_d , \bar{T} AND \bar{W}
(Pages 1.12 through 1.37)

TABLE 1.5-1
VALUES OF $\bar{\lambda}$, $\bar{\gamma}$ AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_D , \bar{T} AND \bar{W}

$\bar{T}_D = 0.$				$\bar{T}_D = 0.$			
\bar{T}	$\bar{\gamma}$	\bar{M}	$\bar{\lambda}$	\bar{T}	$\bar{\gamma}$	\bar{M}	$\bar{\lambda}$
COMPRESSIVE END LOADS				COMPRESSIVE END LOADS			
0.	0.	0.	0.	0.4057143E-00	0.6250000E 00	0.1500000E 01	0.
0.1500000E-01	0.	0.	0.	0.5381333E 00	0.6487420E 00	0.1558387E 01	-0.3000000E-00
0.2666667E-01	0.	0.	0.	0.5820746E 00	0.6684880E 00	0.1606958E 01	-0.4000000E-00
0.4166667E-01	0.	0.	0.	0.6430699E 00	0.6957085E 00	0.1673927E 01	-0.5000000E-00
0.6000000E-01	0.	0.	0.	0.7258127E 00	0.7321367E 00	0.1765569E 01	-0.6000000E-00
0.8166667E-01	0.	0.	0.	0.8378860E 00	0.7804156E 00	0.1882406E 01	-0.7000000E-00
0.1066667E-00	0.	0.	0.	0.9721159E 00	0.8446597E 00	0.2040582E 01	-0.8000000E-00
0.1350000E-00	0.	0.	0.	0.1211592E 01	0.9315309E 00	0.2254540E 01	-0.9000000E-00
0.1666667E-00	0.	0.	0.	0.1539828E 01	0.1052447E 01	0.2525447E 01	-0.1000000E 01
0.2016667E-00	0.	0.	0.	0.2071176E 01	0.1278614E 01	0.2986622E 01	-0.1100000E 01
0.2400000E-00	0.	0.	0.	0.3041085E 01	0.1504201E 01	0.3666049E 01	-0.1200000E 01
0.2816667E-00	0.	0.	0.	0.5735381E 01	0.1988727E 01	0.4860496E 01	-0.1300000E 01
0.3266667E-00	0.	0.	0.	0.1181068E 02	0.3048531E 01	0.7474729E 01	-0.1400000E 01
0.3529167E-00	0.	0.	0.	0.2255509E 02	0.4739773E 01	0.1041412E 02	-0.1450000E 01
0.3750000E-00	0.	0.	0.	0.6293117E 02	0.7110123E 01	0.1751578E 02	-0.1500000E 01
0.4004167E-00	0.	0.	0.	0.7023224E 03	0.2395009E 02	0.5879905E 02	-0.155000E 01
TENSILE END LOADS				TENSILE END LOADS			
-0.1500000E-01	0.	0.	0.	0.4375623E-00	0.6029302E 00	0.1445736E 01	0.3000000E-00
-0.2666667E-01	0.	0.	0.	0.4016713E-00	0.5868110E 00	0.1406110E 01	0.4000000E-00
-0.4166667E-01	0.	0.	0.	0.3587850E-00	0.5673065E 00	0.1358173E 01	0.5000000E-00
-0.6000000E-01	0.	0.	0.	0.3092020E-00	0.5451549E 00	0.1303134E 01	0.6000000E-00
-0.8166667E-01	0.	0.	0.	0.2564499E-00	0.5211012E 00	0.1246660E 01	0.7000000E-00
-0.1066667E-00	0.	0.	0.	0.1996001E-00	0.4795890E-00	0.1182671E 01	0.8000000E-00
-0.1350000E-00	0.	0.	0.	0.1400000E-00	0.4709258E-00	0.1124957E 01	0.9000000E-00
-0.1666667E-00	0.	0.	0.	0.1000000E 01	0.4441628E 00	0.1055837E 01	0.1000000E 01
-0.2016667E-00	0.	0.	0.	0.1101000E 01	0.4186087E 00	0.9953860E 00	0.1100000E 01
-0.2400000E-00	0.	0.	0.	0.1200000E 01	0.3937325E 00	0.9327372E 00	0.1200000E 01
-0.2816667E-00	0.	0.	0.	0.1303000E 01	0.3701328E 00	0.8754756E 00	0.1300000E 01
-0.3266667E-00	0.	0.	0.	0.1403000E 01	0.3478505E 00	0.8189696E 00	0.1400000E 01
-0.3529167E-00	0.	0.	0.	0.1500000E 01	0.3259828E 00	0.7665306E 00	0.1500000E 01
-0.3750000E-00	0.	0.	0.	0.1600000E 01	0.3096274E 00	0.7172131E 00	0.1600000E 01
-0.4004167E-00	0.	0.	0.	0.1700000E 01	0.2985967E 00	0.6710379E 00	0.1700000E 01
-0.4816667E-00	0.	0.	0.	0.1800000E 01	0.2891772E 00	0.6279586E 00	0.1800000E 01
-0.5400000E-00	0.	0.	0.	0.1900000E 01	0.2815618E 00	0.5878719E 00	0.1900000E 01
-0.6066667E-00	0.	0.	0.	0.2000000E 01	0.2759263E 00	0.5508436E 00	0.2000000E 01
-0.6816667E-00	0.	0.	0.	0.2100000E 01	0.2727780E 00	0.5161262E 00	0.2100000E 01
-0.8066667E-00	0.	0.	0.	0.2200000E 01	0.2754671E 00	0.4841416E 00	0.2200000E 01
-0.8816667E-00	0.	0.	0.	0.2300000E 01	0.2812424E 00	0.4545263E 00	0.2300000E 01
-0.9600000E-00	0.	0.	0.	0.2400000E 01	0.2896257E 00	0.4275106E 00	0.2400000E 01
-0.1041667E 01	0.	0.	0.	0.2500000E 01	0.3002714E 01	0.4037258E 00	0.2500000E 01
-0.1126667E 01	0.	0.	0.	0.2600000E 01	0.3128098E 01	0.3822544E 00	0.2600000E 01
-0.1215000E 01	0.	0.	0.	0.2700000E 01	0.3271692E 01	0.3626464E 00	0.2700000E 01
-0.1306667E 01	0.	0.	0.	0.2800000E 01	0.3437876E 01	0.3436282E 00	0.2800000E 01
-0.1401667E 01	0.	0.	0.	0.2900000E 01	0.3621681E 01	0.3258111E 00	0.2900000E 01
-0.1500000E 01	0.	0.	0.	0.3000000E 01	0.3827478E 01	0.3092724E 00	0.3000000E 01
COMPRESSIVE END LOADS				COMPRESSIVE END LOADS			
0.742857E 01	0.1750000E 01	0.1003000E 01	0.	0.4471420E 01	0.1875200E 01	0.4550000E 01	0.
0.2107533E 01	0.1277584E 01	0.3116774E 01	-0.3000000E-00	0.5025127E 01	0.1946224E 01	0.4675160E 01	0.1000000E 01
0.2248279E 01	0.1316976E 01	0.3213916E 01	-0.4000000E-00	0.5505323E 01	0.2005464E 01	0.4805074E 01	0.2000000E 01
0.2447280E 01	0.1391417E 01	0.3447354E 01	-0.5000000E-00	0.5954295E 01	0.2087125E 01	0.4952708E 01	0.3000000E 01
0.2723251E 01	0.1642731E 01	0.3527138E 01	-0.6000000E-00	0.6682310E 01	0.2196410E 01	0.5232708E 01	0.4000000E 01
0.3106547E 01	0.1560851E 01	0.3765402E 01	-0.7000000E-00	0.6887664E 01	0.2341274E 01	0.5684721E 01	0.5000000E 01
0.3688688E 01	0.1588919E 01	0.4081164E 01	-0.8000000E-00	0.8071716E 01	0.2517774E 01	0.6317174E 01	0.6000000E 01
0.4448580E 01	0.1861062E 01	0.4509380E 01	-0.9000000E-00	0.9822550E 01	0.2774974E 01	0.7164620E 01	0.7000000E 01
0.5659511E 01	0.2104894E 01	0.5105894E 01	-1.000000E 01	0.1252513E 02	0.3305000E 01	0.8493867E 01	0.8000000E 01
0.6820681E 01	0.2457227E 01	0.5973354E 01	-1.100000E 01	0.1592571E 02	0.3959841E 01	0.1029115E 02	0.9000000E 01
0.8114438E 02	0.3008401E 01	0.7132098E 01	-1.200000E 01	0.2586975E 02	0.6512674E 01	0.1694915E 02	0.1000000E 02
0.1961075E 02	0.4977453E 01	0.9721096E 01	-1.300000E 01	0.4454113E 02	0.1592618E 01	0.3466494E 02	0.1100000E 02
0.6262711E 02	0.6977453E 01	0.1494246E 02	-1.400000E 01	0.7402291E 02	0.2164924E 01	0.5223519E 02	0.1200000E 02
0.8916910E 02	0.8477545E 01	0.2082074E 02	-1.500000E 01	0.1205192E 03	0.2721332E 02	0.7126457E 02	0.1300000E 02
0.2505997E 03	0.1423625E 02	0.3503155E 02	-1.600000E 01	0.2053586E 03	0.2735543E 02	0.9264237E 02	0.1400000E 02
0.2808088E 03	0.4770018E 02	0.1175997E 03	-1.700000E 01	0.3811229E 03	0.7155271E 02	0.1751111E 03	0.1500000E 02
TENSILE END LOADS				TENSILE END LOADS			
0.1795249E 01	0.1265860E 01	0.2891471E 01	0.3000000E 00	0.4058111E 01	0.1809771E 01	0.4551329E 01	0.4000000E 01
0.1686605E 01	0.1175622E 01	0.2812220E 01	0.4000000E 00	0.3628437E 01	0.1728043E 01	0.4210134E 01	0.5000000E 01
0.1560140E 01	0.1134613E 01	0.2716347E 01	0.5000000E 00	0.3364424E 01	0.1701724E 01	0.4074522E 01	0.6000000E 01
0.1439681E 01	0.1090110E 01	0.2607408E 01	0.6000000E 00	0.3269240E 01	0.1655465E 01	0.3941314E 01	0.7000000E 01
0.1270800E 01	0.1042020E 01	0.2489323E 01	0.7000000E 00	0.2221138E 01	0.1563504E 01	0.3737206E 01	0.8000000E 01
0.1118401E 01	0.9916081E 00	0.2365313E 01	0.8000000E 00	0.2662714E 01	0.1607547E 01	0.3551710E 01	0.9000000E 01
0.9662964E 00	0.9400515E 00	0.2238558E 01	0.9000000E 00	0.2342927E 01	0.1610027E 01	0.3378754E 01	0.1000000E 02
0.8172695E 00	0.8881256E 00	0.2111674E 01	0.1000000E 01	0.2047169E 01	0.1532488E 01	0.3216751E 01	0.2000000E 02
0.6711270E 00	0.8371775E 00	0.1985773E 01	0.1100000E 01	0.1766819E 01	0.1456066E 01	0.3072916E 01	0.3000000E 02
0.5346651E 00	0.7871065E 00	0.1864974E 01	0.1200000E 01	0.1504640E 01	0.1381799E 01	0.2929811E 01	0.4000000E 02
0.4028452E 00	0.7402756E 00	0.1748971E 01	0.1300000E 01	0.1258491E 01	0.1310749E 01	0.2787477E 01	0.5000000E 02
0.2769189E 00	0.6949011E 00	0.1637394E 01	0.1400000E 01	0.1031431E 01	0.1240532E 01	0.2659464E 01	0.6000000E 02
0.1566888E 00	0.6519111E 00	0.1535077E 01	0.1500000E 01	0.8212101E 00	0.1177948E 01	0.2549816E 01	0.7000000E 02
0.5149075E-01	0.6115523E 00	0.1437424E 01	0.1600000E 01	0.6266884E 00	0.2173215E 00	0.2451419E 01	0.8000000E 02
-0.6918487E-01	0.5716766E 00	0.1332075E 01	0.1700000E 01	0.4452674E 00	0.2031312E 00	0.2310112E 01	0.9000000E 02
-0.1767115E-00	0.5328297E 00	0.1235917E 01	0.1800000E 01	0.2773211E-00	0.2077459E 00	0.1888877E 01	0.1000000E 03
-0.2812458E-00	0.5053330E 00	0.1137570E 01	0.1900000E 01	0.1192802E-00	0.2175225E 00	0.1761627E 01	0.2000000E 03
-0.3810703E-00	0.4766758E 00	0.1041292E 01	0.2000000E 01	0.3003341E 01	0.2120130E 00	0.1651945E 01	0.3000000E 03
-0.4847527E-00	0.4462017E 00	0.1032252E 01	0.2100000E 01	0.3719924E 01	0.2069312E 00	0.1558017E 01	0.4000000E 03
-0.5899862E-00	0.4157178E-00	0.9862028E 00	0.2200000E 01	0.4307885E 01	0.2024844E 00	0.1452424E 01	0.5000000E 03
-0.6899872E-00	0.3857167E-00	0.9090488E 00	0.2300000E 01	0.4768641E 01	0.1982474E 00	0.1353573E 01	0.6000000E 03
-0.7850948E-00	0.3575124E-00	0.8542116E 00	0.2400000E 01	0.5086841E 01	0.1943821E 00	0.1261120E 01	0.7000000E 03
-0.8858790E-00	0.3314477E-00	0.8034516E 00	0.2500000E 01	0.6910950E 01	0.2271716E 00	0.1205177E 01	0.8000000E 03
-0.9875901E-00	0.3118860E-00	0.7564507E 00	0.2600000E 01	0.8137460E 01	0.2497290E 00	0.1134474E 01	0.9000000E 03
-0.1090610E 01	0.3137289E-00	0.7129131E 00	0.2700000E 01	0.9155121E 01	0.2705931E 00	0.1069179E 01	0.1000000E 04
-0.1195181E 01	0.2968658E-00	0.6725724E 00	0.2800000E 01	0.1055884E 01	0.2942981E 00	0.1003974E 01	0.2000000E 04
-0.1301552E 01	0.2819138E-00	0.6351622E 00	0.2900000E 01	0.1176409E 01	0.3217901E 00	0.9527819E 00	0.3000000E 04
-0.1409912E 01	0.2666189E-00	0.6004480E 00	0.3000000E 01	0.1297302E 01	0.3499251E 00	0.9006721E 00	0.4000000E 04

TABLE 1.5-1 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_D , \bar{T} AND \bar{W}

\bar{T}_D	0.	\bar{W}	0.1200000E 02	\bar{T}_D	0.	\bar{W}	0.1500000E 02
\bar{T}	\bar{y}	\bar{M}	$\bar{\lambda}$	\bar{T}	\bar{y}	\bar{M}	$\bar{\lambda}$
COMPRESSIVE END LOADS				COMPRESSIVE END LOADS			
0.7777777E 01	0.2500000E 01	0.6000000E 01	0.	0.4286E 02	0.3125000E 01	0.7500000E 01	0.
0.8385E 01	0.2599999E 01	0.6233597E 01	-0.1000000E -00	0.1309391E 02	0.3263710E 01	0.7791936E 01	-0.1000000E -00
0.8913194E 01	0.2673952E 01	0.6427832E 01	-0.4000000E -00	0.1391189E 02	0.3342440E 01	0.8036790E 01	-0.4000000E -00
0.9466118E 01	0.2728384E 01	0.6695703E 01	-0.5000000E -00	0.1507670E 02	0.3478542E 01	0.8169636E 01	-0.5000000E -00
0.1071300E 02	0.2928547E 01	0.7054277E 01	-0.6000000E -00	0.1670530E 02	0.3660681E 01	0.8617886E 01	-0.6000000E -00
0.1218119E 02	0.3121662E 01	0.7529615E 01	-0.7000000E -00	0.1898718E 02	0.3902076E 01	0.9412018E 01	-0.7000000E -00
0.1427387E 02	0.3378619E 01	0.8162529E 01	-0.8000000E -00	0.2224291E 02	0.4273799E 01	0.1020291E 02	-0.8000000E -00
0.1735728E 02	0.3726124E 01	0.9018160E 01	-0.9000000E -00	0.2704481E 02	0.4657654E 01	0.1127270E 02	-0.9000000E -00
0.2213724E 02	0.4209789E 01	0.1027779E 02	-0.1000000E 01	0.3469569E 02	0.5262716E 01	0.1276224E 02	-0.1000000E 01
0.1012327E 02	0.4918454E 01	0.119404E 02	-0.1100000E 01	0.4695417E 02	0.6181068E 01	0.1493111E 02	-0.1100000E 01
0.4505716E 02	0.6016803E 01	0.1466420E 02	-0.1200000E 01	0.7026712E 02	0.7521006E 01	0.1835024E 02	-0.1200000E 01
0.7853800E 02	0.7945907E 01	0.1944319E 02	-0.1300000E 01	0.1225888E 03	0.9943633E 01	0.2436044E 02	-0.1300000E 01
0.1840708E 03	0.1219333E 02	0.2992627E 02	-0.1400000E 01	0.2874269E 03	0.1524166E 02	0.3737364E 02	-0.1400000E 01
0.3562522E 03	0.1695909E 02	0.4155649E 02	-0.1500000E 01	0.5558672E 03	0.2119886E 02	0.5107581E 02	-0.1500000E 01
0.1001274E 04	0.2847249E 02	0.7026311E 02	-0.1500000E 01	0.1564279E 04	0.3559062E 02	0.8157889E 02	-0.1500000E 01
0.1123115E 04	0.3542017E 02	0.2351929E 03	-0.1500000E 01	0.1754845E 05	0.1192505E 03	0.2933992E 03	-0.1500000E 01
TENSILE END LOADS				TENSILE END LOADS			
0.7225997E 01	0.2411721E 01	0.5782945E 01	0.1000000E -00	0.1128912E 02	0.3014651E 01	0.7226681E 01	0.1000000E -00
0.6826740E 01	0.2347244E 01	0.5624441E 01	0.4000000E -00	0.1068180E 02	0.2914055E 01	0.7030551E 01	0.4000000E -00
0.6165360E 01	0.2269728E 01	0.5432693E 01	0.5000000E -00	0.9969640E 01	0.2816531E 01	0.6790667E 01	0.5000000E -00
0.5858724E 01	0.21860420E 01	0.5214977E 01	0.6000000E -00	0.9188004E 01	0.2725775E 01	0.6514873E 01	0.6000000E -00
0.5328199E 01	0.2081005E 01	0.4978064E 01	0.7000000E -00	0.8171243E 01	0.2605506E 01	0.6223502E 01	0.7000000E -00
0.4793602E 01	0.1983390E 01	0.4733626E 01	0.8000000E -00	0.7500004E 01	0.2479245E 01	0.5913281E 01	0.8000000E -00
0.4270191E 01	0.1880103E 01	0.4477116E 01	0.9000000E -00	0.6748115E 01	0.2350127E 01	0.5583596E 01	0.9000000E -00
0.3769078E 01	0.1776651E 01	0.4223349E 01	0.1000000E 01	0.5982931E 01	0.2220914E 01	0.5279186E 01	0.1000000E 01
0.3247508E 01	0.1674755E 01	0.3973547E 01	0.1100000E 01	0.5265794E 01	0.2093444E 01	0.4966933E 01	0.1100000E 01
0.2859463E 01	0.1575730E 01	0.3730949E 01	0.1200000E 01	0.4602910E 01	0.1964966E 01	0.4643686E 01	0.1200000E 01
0.2456147E 01	0.1480533E 01	0.3497402E 01	0.1300000E 01	0.3996470E 01	0.1850666E 01	0.4327378E 01	0.1300000E 01
0.2087675E 01	0.1389802E 01	0.3275286E 01	0.1400000E 01	0.3445743E 01	0.1737253E 01	0.4009495E 01	0.1400000E 01
0.1751721E 01	0.1303931E 01	0.3066154E 01	0.1500000E 01	0.2948005E 01	0.1629914E 01	0.3682693E 01	0.1500000E 01
0.1445961E 01	0.1223105E 01	0.2868052E 01	0.1600000E 01	0.2499314E 01	0.1528081E 01	0.3380065E 01	0.1600000E 01
0.1167460E 01	0.1147153E 01	0.2684149E 01	0.1700000E 01	0.2095394E 01	0.1444101E 01	0.3155187E 01	0.1700000E 01
0.9131540E 00	0.1076794E 01	0.2511814E 01	0.1800000E 01	0.1730553E 01	0.1365743E 01	0.2939772E 01	0.1800000E 01
0.6800167E 00	0.1010466E 01	0.2351184E 01	0.1900000E 01	0.1400684E 01	0.1281533E 01	0.27491570E 01	0.1900000E 01
0.4651786E 00	0.9495117E 00	0.2212593E 01	0.2000000E 01	0.1101842E 01	0.1186649E 01	0.2553242E 01	0.2000000E 01
0.2659913E -00	0.8924026E 00	0.2064503E 01	0.2100000E 01	0.8790486E 00	0.1115503E 01	0.2360631E 01	0.2100000E 01
0.8005517E -01	0.8395246E 00	0.1936455E 01	0.2200000E 01	0.5786162E 00	0.1049444E 01	0.2262707E 01	0.2200000E 01
-0.9477098E -01	0.7905270E 00	0.1818097E 01	0.2300000E 01	0.3478578E -00	0.9881623E 00	0.2272671E 01	0.2300000E 01
-0.2603793E -00	0.7450647E 00	0.1708477E 01	0.2400000E 01	0.1531574E -00	0.9331509E 00	0.2155534E 01	0.2400000E 01
-0.4184702E -00	0.7028955E 00	0.1606903E 01	0.2500000E 01	-0.4785459E -01	0.8646191E 00	0.2000529E 01	0.2500000E 01
-0.5705631E -00	0.6617720E 00	0.1512701E 01	0.2600000E 01	-0.2574627E -00	0.8297150E 00	0.1902117E 01	0.2600000E 01
-0.7174592E -00	0.6224578E 00	0.1427613E 01	0.2700000E 01	-0.4375415E -00	0.7883223E 00	0.1782291E 01	0.2700000E 01
-0.8407417E -00	0.5847115E 00	0.1344545E 01	0.2800000E 01	-0.6094038E -00	0.7421644E 00	0.1681431E 01	0.2800000E 01
-0.1001208E 01	0.5423871E 00	0.1270374E 01	0.2900000E 01	-0.7154908E -00	0.7029819E 00	0.1587906E 01	0.2900000E 01
-0.1194648E 01	0.5112118E 00	0.1200647E 01	0.3000000E 01	-0.8144508E -00	0.6664221E 00	0.1501170E 01	0.3000000E 01
TENSILE END LOADS				TENSILE END LOADS			
0.1746571E 02	0.3750000E 01	0.9000000E 01	0.	0.2380000E 02	0.4575000E 01	0.1050000E 02	0.
0.1884011E 02	0.3902452E 01	0.9150517E 01	-0.1000000E -00	0.2504645E 02	0.4551119E 01	0.1090007E 02	-0.1000000E -00
0.2002129E 02	0.4019923E 01	0.9264174E 01	-0.4000000E -00	0.2724140E 02	0.4677414E 01	0.1124071E 02	-0.4000000E -00
0.2169218E 02	0.4174551E 01	0.1003556E 02	-0.5000000E -00	0.2951035E 02	0.4809959E 01	0.1171749E 02	-0.5000000E -00
0.2402924E 02	0.4392782E 01	0.1058142E 02	-0.6000000E -00	0.3248479E 02	0.5124957E 01	0.1214440E 02	-0.6000000E -00
0.2730559E 02	0.4662444E 01	0.1123442E 02	-0.7000000E -00	0.3713345E 02	0.5462990E 01	0.1317483E 02	-0.7000000E -00
0.3199287E 02	0.5067938E 01	0.1223444E 02	-0.8000000E -00	0.4344211E 02	0.5912418E 01	0.1424080E 02	-0.8000000E -00
0.3800512E 02	0.5598718E 01	0.1351405E 02	-0.9000000E -00	0.5207921E 02	0.6520718E 01	0.1570718E 02	-0.9000000E -00
0.4590047E 02	0.6314608E 01	0.1531405E 02	-0.1000000E 01	0.6347115E 02	0.7338113E 01	0.1786713E 02	-0.1000000E 01
0.5752328E 02	0.7371681E 01	0.1791273E 02	-0.1100000E 01	0.7814348E 02	0.8402094E 01	0.2096243E 02	-0.1100000E 01
0.1010791E 03	0.9025204E 01	0.2199472E 02	-0.1200000E 01	0.9144732E 02	0.1027440E 02	0.2566234E 02	-0.1200000E 01
0.1764044E 03	0.1191216E 02	0.2915449E 02	-0.1300000E 01	0.2400000E 03	0.1392710E 02	0.3432644E 02	-0.1300000E 01
0.3137510E 03	0.1878399E 02	0.4480610E 02	-0.1400000E 01	0.4640442E 03	0.2115812E 02	0.5212111E 02	-0.1400000E 01
0.7997186E 03	0.2543864E 02	0.6240473E 02	-0.1500000E 01	0.1060179E 04	0.2907441E 02	0.7249683E 02	-0.1500000E 01
0.2252397E 04	0.3272874E 02	0.1050297E 03	-0.1600000E 01	0.1553528E 04	0.4097260E 02	0.1226104E 03	-0.1600000E 01
0.2524959E 04	0.3431005E 03	0.1527291E 03	-0.1500000E 01	0.1619454E 04	0.4365756E 03	0.1615498E 03	-0.1500000E 01
TENSILE END LOADS				TENSILE END LOADS			
0.1627745E 02	0.3417582E 01	0.8674441E 01	0.1000000E -00	0.2216162E 02	0.4178012E 01	0.1012015E 02	0.1000000E -00
0.1519149E 02	0.3420860E 01	0.8643661E 01	0.4000000E -00	0.1529192E 02	0.4107477E 01	0.9847721E 01	0.4000000E -00
0.1407461E 02	0.3403597E 01	0.8614000E 01	0.5000000E -00	0.1273053E 02	0.3971146E 01	0.9507214E 01	0.5000000E -00
0.1325712E 02	0.3370910E 01	0.8572445E 01	0.6000000E -00	0.1050616E 02	0.3816785E 01	0.9126239E 01	0.6000000E -00
0.1209053E 02	0.3328807E 01	0.8517342E 01	0.7000000E -00	0.0846305E 02	0.3647170E 01	0.8712743E 01	0.7000000E -00
0.1091894E 02	0.2973044E 01	0.7505940E 01	0.8000000E -00	0.0670041E 02	0.3470943E 01	0.8274561E 01	0.8000000E -00
0.9776807E 01	0.2820155E 01	0.6715675E 01	0.9000000E -00	0.1335720E 02	0.3781974E 01	0.7814974E 01	0.9000000E -00
0.6688150E 01	0.2564977E 01	0.6135023E 01	0.1000000E 01	0.1026654E 02	0.4109101E 01	0.7320460E 01	0.1000000E 01
0.7671476E 01	0.2512132E 01	0.5940120E 01	0.1100000E 01	0.0951454E 02	0.4293021E 01	0.6951704E 01	0.1100000E 01
0.6713741E 01	0.2361925E 01	0.5594623E 01	0.1200000E 01	0.0925704E 02	0.4575726E 01	0.6529160E 01	0.1200000E 01
0.5878650E 01	0.2220797E 01	0.5246584E 01	0.1300000E 01	0.0840754E 02	0.4870754E 01	0.6121329E 01	0.1300000E 01
0.5105601E 01	0.2084701E 01	0.4911802E 01	0.1400000E 01	0.0784754E 02	0.5181254E 01	0.5712579E 01	0.1400000E 01
0.4410127E 01	0.1955027E 01	0.4559232E 01	0.1500000E 01	0.0741090E 02	0.5512188E 01	0.5348771E 01	0.1500000E 01
0.3786745E 01	0.1834657E 01	0.4303217E 01	0.1600000E 01	0.0693255E 02	0.5850451E 01	0.5020441E 01	0.1600000E 01
0.3228809E 01	0.1721050E 01	0.4072624E 01	0.1700000E 01	0.0648478E 02	0.6202768E 01	0.4697261E 01	0.1700000E 01
0.2729496E 01	0.1618472E 01	0.3767175E 01	0.1800000E 01	0.0607024E 02	0.6569340E 01	0.4395739E 01	0.1800000E 01
0.2282121E 01	0.1515999E 01	0.3427244E 01	0.1900000E 01	0.0571466E 02	0.6954866E 01	0.4135117E 01	0.1900000E 01
0.1879983E 01	0.1424027E 01	0.3101890E 01	0.2000000E 01	0.05409410E 02	0.7364136E 01	0.3905531E 01	0.2000000E 01
0							

VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_d , T AND W [illegible]

VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_d , \bar{T} AND \bar{W}

\bar{T}_D	\bar{T}	\bar{W}	\bar{V}	\bar{T}_D	\bar{T}	\bar{W}	\bar{V}
0.00000000 01	0.00000000 01	0.00000000 01	0.00000000 01	0.00000000 01	0.00000000 01	0.00000000 01	0.00000000 01
COMPRESSIVE END LOADS				COMPRESSIVE END LOADS			
0.07333333 01	0.21500000 01	0.16533333 02	0.10819058 02	0.07333333 01	0.21500000 01	0.16533333 02	0.10819058 02
0.12618878 01	0.26833333 01	0.19121701 02	0.11662710 02	0.12618878 01	0.26833333 01	0.19121701 02	0.11662710 02
0.17201193 01	0.29566666 01	0.21059594 02	0.12572578 02	0.17201193 01	0.29566666 01	0.21059594 02	0.12572578 02
0.21888877 01	0.26190000 01	0.21115751 02	0.13678058 02	0.21888877 01	0.26190000 01	0.21115751 02	0.13678058 02
0.26738171 01	0.27733333 01	0.21252987 02	0.14875876 02	0.26738171 01	0.27733333 01	0.21252987 02	0.14875876 02
0.30500000 01	0.29500000 01	0.21296674 02	0.15700000 02	0.30500000 01	0.29500000 01	0.21296674 02	0.15700000 02
0.32947661 02	0.31718678 01	0.21552746 02	0.16720700 02	0.32947661 02	0.31718678 01	0.21552746 02	0.16720700 02
0.35002172 02	0.35166666 01	0.21566666 02	0.17000000 02	0.35002172 02	0.35166666 01	0.21566666 02	0.17000000 02
0.39117681 02	0.36366666 01	0.21566666 02	0.17000000 02	0.39117681 02	0.36366666 01	0.21566666 02	0.17000000 02
0.43000000 02	0.40000000 01	0.21626666 02	0.17125000 02	0.43000000 02	0.40000000 01	0.21626666 02	0.17125000 02
0.45000000 02	0.45000000 01	0.21650000 02	0.17250000 02	0.45000000 02	0.45000000 01	0.21650000 02	0.17250000 02
0.46767691 02	0.47500000 01	0.21711101 02	0.17400000 02	0.46767691 02	0.47500000 01	0.21711101 02	0.17400000 02
0.49000000 03	0.51172222 01	0.21722222 02	0.17400000 02	0.49000000 03	0.51172222 01	0.21722222 02	0.17400000 02
0.50000000 03	0.51717171 02	0.21717171 02	0.17400000 02	0.50000000 03	0.51717171 02	0.21717171 02	0.17400000 02
0.55152755 03	0.52667269 01	0.21703333 02	0.17500000 02	0.55152755 03	0.52667269 01	0.21703333 02	0.17500000 02
0.56676701 04	0.56500000 02	0.21722222 02	0.17500000 02	0.56676701 04	0.56500000 02	0.21722222 02	0.17500000 02
TENSILE END LOADS				TENSILE END LOADS			
0.02772833 01	0.22222222 01	0.16222222 02	0.10819058 02	0.02772833 01	0.22222222 01	0.16222222 02	0.10819058 02
0.07333333 01	0.22222222 01	0.16222222 02	0.10819058 02	0.07333333 01	0.22222222 01	0.16222222 02	0.10819058 02
0.12618878 01	0.26833333 01	0.19121701 02	0.11662710 02	0.12618878 01	0.26833333 01	0.19121701 02	0.11662710 02
0.17201193 01	0.29566666 01	0.21059594 02	0.12572578 02	0.17201193 01	0.29566666 01	0.21059594 02	0.12572578 02
0.21888877 01	0.26190000 01	0.21115751 02	0.13678058 02	0.21888877 01	0.26190000 01	0.21115751 02	0.13678058 02
0.26738171 01	0.27733333 01	0.21252987 02	0.14875876 02	0.26738171 01	0.27733333 01	0.21252987 02	0.14875876 02
0.30500000 01	0.29500000 01	0.21296674 02	0.15700000 02	0.30500000 01	0.29500000 01	0.21296674 02	0.15700000 02
0.32947661 02	0.31718678 01	0.21552746 02	0.16720700 02	0.32947661 02	0.31718678 01	0.21552746 02	0.16720700 02
0.35002172 02	0.35166666 01	0.21566666 02	0.17000000 02	0.35002172 02	0.35166666 01	0.21566666 02	0.17000000 02
0.39117681 02	0.36366666 01	0.21566666 02	0.17000000 02	0.39117681 02	0.36366666 01	0.21566666 02	0.17000000 02
0.43000000 02	0.40000000 01	0.21626666 02	0.17125000 02	0.43000000 02	0.40000000 01	0.21626666 02	0.17125000 02
0.45000000 02	0.45000000 01	0.21650000 02	0.17250000 02	0.45000000 02	0.45000000 01	0.21650000 02	0.17250000 02
0.46767691 02	0.47500000 01	0.21711101 02	0.17400000 02	0.46767691 02	0.47500000 01	0.21711101 02	0.17400000 02
0.49000000 03	0.51172222 01	0.21722222 02	0.17400000 02	0.49000000 03	0.51172222 01	0.21722222 02	0.17400000 02
0.50000000 03	0.51717171 02	0.21717171 02	0.17400000 02	0.50000000 03	0.51717171 02	0.21717171 02	0.17400000 02
0.55152755 03	0.52667269 01	0.21703333 02	0.17500000 02	0.55152755 03	0.52667269 01	0.21703333 02	0.17500000 02
0.56676701 04	0.56500000 02	0.21722222 02	0.17500000 02	0.56676701 04	0.56500000 02	0.21722222 02	0.17500000 02
TENSILE END LOADS				TENSILE END LOADS			
0.02772833 01	0.22222222 01	0.16222222 02	0.10819058 02	0.02772833 01	0.22222222 01	0.16222222 02	0.10819058 02
0.07333333 01	0.22222222 01	0.16222222 02	0.10819058 02	0.07333333 01	0.22222222 01	0.16222222 02	0.10819058 02
0.12618878 01	0.26833333 01	0.19121701 02	0.11662710 02	0.12618878 01	0.26833333 01	0.19121701 02	0.11662710 02
0.17201193 01	0.29566666 01	0.21059594 02	0.12572578 02	0.17201193 01	0.29566666 01	0.21059594 02	0.12572578 02
0.21888877 01	0.26190000 01	0.21115751 02	0.13678058 02	0.21888877 01	0.26190000 01	0.21115751 02	0.13678058 02
0.26738171 01	0.27733333 01	0.21252987 02	0.14875876 02	0.26738171 01	0.27733333 01	0.21252987 02	0.14875876 02
0.30500000 01	0.29500000 01	0.21296674 02	0.15700000 02	0.30500000 01	0.29500000 01	0.21296674 02	0.15700000 02
0.32947661 02	0.31718678 01	0.21552746 02	0.16720700 02	0.32947661 02	0.31718678 01	0.21552746 02	0.16720700 02
0.35002172 02	0.35166666 01	0.21566666 02	0.17000000 02	0.35002172 02	0.35166666 01	0.21566666 02	0.17000000 02
0.39117681 02	0.36366666 01	0.21566666 02	0.17000000 02	0.39117681 02	0.36366666 01	0.21566666 02	0.17000000 02
0.43000000 02	0.40000000 01	0.21626666 02	0.17125000 02	0.43000000 02	0.40000000 01	0.21626666 02	0.17125000 02
0.45000000 02	0.45000000 01	0.21650000 02	0.17250000 02	0.45000000 02	0.45000000 01	0.21650000 02	0.17250000 02
0.46767691 02	0.47500000 01	0.21711101 02	0.17400000 02	0.46767691 02	0.47500000 01	0.21711101 02	0.17400000 02
0.49000000 03	0.51172222 01	0.21722222 02	0.17400000 02	0.49000000 03	0.51172222 01	0.21722222 02	0.17400000 02
0.50000000 03	0.51717171 02	0.21717171 02	0.17400000 02	0.50000000 03	0.51717171 02	0.21717171 02	0.17400000 02
0.55152755 03	0.52667269 01	0.21703333 02	0.17500000 02	0.55152755 03	0.52667269 01	0.21703333 02	0.17500000 02
0.56676701 04	0.56500000 02	0.21722222 02	0.17500000 02	0.56676701 04	0.56500000 02	0.21722222 02	0.17500000 02
TENSILE END LOADS				TENSILE END LOADS			
0.02772833 01	0.22222222 01	0.16222222 02	0.10819058 02	0.02772833 01	0.22222222 01	0.16222222 02	0.10819058 02
0.07333333 01	0.22222222 01	0.16222222 02	0.10819058 02	0.07333333 01	0.22222222 01	0.16222222 02	0.10819058 02
0.12618878 01	0.26833333 01	0.19121701 02	0.11662710 02	0.12618878 01	0.26833333 01	0.19121701 02	0.11662710 02
0.17201193 01	0.29566666 01	0.21059594 02	0.12572578 02	0.17201193 01	0.29566666 01	0.21059594 02	0.12572578 02
0.21888877 01	0.26190000 01	0.21115751 02	0.13678058 02	0.21888877 01	0.26190000 01	0.21115751 02	0.13678058 02
0.26738171 01	0.27733333 01	0.21252987 02	0.14875876 02	0.26738171 01	0.27733333 01	0.21252987 02	0.14875876 02
0.30500000 01	0.29500000 01	0.21296674 02	0.15700000 02	0.30500000 01	0.29500000 01	0.21296674 02	0.15700000 02
0.32947661 02	0.31718678 01	0.21552746 02	0.16720700 02	0.32947661 02	0.31718678 01	0.21552746 02	0.16720700 02
0.35002172 02	0.35166666 01	0.21566666 02	0.17000000 02	0.35002172 02	0.35166666 01	0.21566666 02	0.17000000 02
0.39117681 02	0.36366666 01	0.21566666 02	0.17000000 02	0.39117681 02	0.36366666 01	0.21566666 02	0.17000000 02
0.43000000 02	0.40000000 01	0.21626666 02	0.17125000 02	0.43000000 02	0.40000000 01	0.21626666 02	0.17125000 02
0.45000000 02	0.45000000 01	0.21650000 02	0.17250000 02	0.45000000 02	0.45000000 01	0.21650000 02	0.17250000 02
0.46767691 02	0.47500000 01	0.21711101 02	0.17400000 02	0.46767691 02	0.47500000 01	0.21711101 02	0.17400000 02
0.49000000 03	0.51172222 01	0.21722222 02	0.17400000 02	0.49000000 03	0.51172222 01	0.21722222 02	0.17400000 02
0.50000000 03	0.51717171 02	0.21717171 02	0.17400000 02	0.50000000 03	0.51717171 02	0.21717171 02	0.17400000 02
0.55152755 03	0.52667269 01	0.21703333 02	0.17500000 02	0.55152755 03	0.52667269 01	0.21703333 02	0.17500000 02
0.56676701 04	0.56500000 02	0.21722222 02	0.17500000 02	0.56676701 04	0.56500000 02	0.21722222 02	0.17500000 02
TENSILE END LOADS				TENSILE END LOADS			
0.02772833 01	0.22222222 01	0.16222222 02	0.10819058 02	0.02772833 01	0.22222222 01	0.16222222 02	0.10819058 02
0.07333333 01	0.22222222 01	0.16222222 02	0.10819058 02	0.07333333 01	0.22222222 01	0.16222222 02	0.10819058 02
0.12618878 01	0.26833333 01	0.19121701 02	0.11662710 02	0.12618878 01	0.26833333 01	0.19121701 02	0.11662710 02
0.17201193 01	0.29566666 01	0.21059594 02	0.12572578 02	0.17201193 01	0.29566666 01	0.21059594 02	0.12572578 02
0.21888877 01	0.26190000 01	0.21115751 02	0.13678058 02	0.21888877 01	0.26190000 01	0.21115751 02	0.13678058 02
0.26738171 01	0.27733333 01	0.21252987 02	0.14875876 02	0.26738171 01	0.27733333 01	0.21252987 02	0.14875876 02
0.30500000 01	0.29500000 01	0.21296674 02	0.15700000 02	0.30500000 01	0.29500000 01	0.21296674 02	0.15700000 02
0.32947661 02	0.31718678 01	0.21552746 02	0.16720700 02	0.32947661 02	0.31718678 01	0.21552746 02	0.16720700 02
0.35002172 02	0.35166666 01	0.21566666 02	0.17000000 02	0.35002172 02	0.35166666 01	0.21566666 02	0.17000000 02
0.39117681 02	0.36366666 01	0.21566666 02	0.17000000 02	0.39117681 02	0.36366666 01	0.21566666 02	0.17000000 02
0.43000000 02	0.40000000 01	0.21626666 02	0.17125000 02	0.43000000 02	0.40000000 01	0.21626666 02	0.17125000 02
0.45000000 02	0.45000000 01	0.21650000 02	0.17250000 02	0.45000000 02	0.45000000 01	0.21650000 02	0.17250000 02
0.46767691 02	0.47500000 01	0.21711101 02	0.17400000 02	0.46767691 02	0.47500000 01	0.21711101 02	0.17400000 02
0.49000000 03	0.51172222 01	0.21722222 02	0.17400000 02	0.49000000 03	0.51172222 01	0.21722222 02	0.17

TABLE 1.5-1 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_d , \bar{T} AND \bar{W}

0.1200000E 02				0.3000000E 02				0.1200000E 02				0.6000000E 02			
W				W				W				W			
M				M				M				M			
COMPRESSION END LOADS				COMPRESSION END LOADS				COMPRESSION END LOADS				COMPRESSION END LOADS			
0.3888571E 02	0.5375000E 01	0.1500000E 01	0.	0.3888571E 02	0.5375000E 01	0.1500000E 01	0.	0.3888571E 02	0.5375000E 01	0.1500000E 01	0.	0.3888571E 02	0.5375000E 01	0.1500000E 01	0.
0.4185295E 02	0.5584805E 01	0.0973876E 00	-0.1000000E -05	0.4185295E 02	0.5584805E 01	0.0973876E 00	-0.1000000E -05	0.4185295E 02	0.5584805E 01	0.0973876E 00	-0.1000000E -05	0.4185295E 02	0.5584805E 01	0.0973876E 00	-0.1000000E -05
0.4440433E 02	0.5759394E 01	0.5705047E 00	-0.0000000E -00	0.4440433E 02	0.5759394E 01	0.5705047E 00	-0.0000000E -00	0.4440433E 02	0.5759394E 01	0.5705047E 00	-0.0000000E -00	0.4440433E 02	0.5759394E 01	0.5705047E 00	-0.0000000E -00
0.4804167E 02	0.6000000E 01	0.	-0.0000000E -00	0.4804167E 02	0.6000000E 01	0.	-0.0000000E -00	0.4804167E 02	0.6000000E 01	0.	-0.0000000E -00	0.4804167E 02	0.6000000E 01	0.	-0.0000000E -00
0.5315357E 02	0.6322140E 01	-0.7759704E 00	-0.6000000E -00	0.5315357E 02	0.6322140E 01	-0.7759704E 00	-0.6000000E -00	0.5315357E 02	0.6322140E 01	-0.7759704E 00	-0.6000000E -00	0.5315357E 02	0.6322140E 01	-0.7759704E 00	-0.6000000E -00
0.5027531E 02	0.6789199E 01	-0.1801707E 01	-0.7000000E 00	0.5027531E 02	0.6789199E 01	-0.1801707E 01	-0.7000000E 00	0.5027531E 02	0.6789199E 01	-0.1801707E 01	-0.7000000E 00	0.5027531E 02	0.6789199E 01	-0.1801707E 01	-0.7000000E 00
0.7047550E 02	-0.7317669E 01	0.1818108E 01	-0.8000000E 00	0.7047550E 02	-0.7317669E 01	0.1818108E 01	-0.8000000E 00	0.7047550E 02	-0.7317669E 01	0.1818108E 01	-0.8000000E 00	0.7047550E 02	-0.7317669E 01	0.1818108E 01	-0.8000000E 00
0.8553552E 02	-0.8086679E 01	-0.6505170E 01	-0.9000000E 00	0.8553552E 02	-0.8086679E 01	-0.6505170E 01	-0.9000000E 00	0.8553552E 02	-0.8086679E 01	-0.6505170E 01	-0.9000000E 00	0.8553552E 02	-0.8086679E 01	-0.6505170E 01	-0.9000000E 00
0.1089272E 03	0.1157748E 01	-0.7857847E 01	-0.1000000E 01	0.1089272E 03	0.1157748E 01	-0.7857847E 01	-0.1000000E 01	0.1089272E 03	0.1157748E 01	-0.7857847E 01	-0.1000000E 01	0.1089272E 03	0.1157748E 01	-0.7857847E 01	-0.1000000E 01
0.1608974E 03	0.1071789E 02	-0.1168891E 02	-0.1100000E 01	0.1608974E 03	0.1071789E 02	-0.1168891E 02	-0.1100000E 01	0.1608974E 03	0.1071789E 02	-0.1168891E 02	-0.1100000E 01	0.1608974E 03	0.1071789E 02	-0.1168891E 02	-0.1100000E 01
0.2213161E 03	0.1315999E 02	-0.1785019E 02	-0.1200000E 01	0.2213161E 03	0.1315999E 02	-0.1785019E 02	-0.1200000E 01	0.2213161E 03	0.1315999E 02	-0.1785019E 02	-0.1200000E 01	0.2213161E 03	0.1315999E 02	-0.1785019E 02	-0.1200000E 01
0.3859326E 03	-0.1785507E 02	0.2279990E 02	-0.1300000E 01	0.3859326E 03	-0.1785507E 02	0.2279990E 02	-0.1300000E 01	0.3859326E 03	-0.1785507E 02	0.2279990E 02	-0.1300000E 01	0.3859326E 03	-0.1785507E 02	0.2279990E 02	-0.1300000E 01
0.9045676E 03	-0.2680504E 02	-0.5312715E 02	-0.1400000E 01	0.9045676E 03	-0.2680504E 02	-0.5312715E 02	-0.1400000E 01	0.9045676E 03	-0.2680504E 02	-0.5312715E 02	-0.1400000E 01	0.9045676E 03	-0.2680504E 02	-0.5312715E 02	-0.1400000E 01
0.1747495E 04	-0.3741707E 02	-0.7716846E 02	-0.1500000E 01	0.1747495E 04	-0.3741707E 02	-0.7									

TABLE 1.5-1 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_d , \bar{T} AND \bar{W}

IMPRESSIONS EN C.A.S.									
1.05133333	2	1.0625	1	1.07375	1	1.08500	2	1.09625	1
1.0625	1	1.07375	1	1.08500	2	1.09625	1	1.10750	1
1.07375	1	1.08500	2	1.09625	1	1.10750	1	1.11875	1
1.08500	2	1.09625	1	1.10750	1	1.11875	1	1.13000	2
1.09625	1	1.10750	1	1.11875	1	1.13000	2	1.14125	1
1.10750	1	1.11875	1	1.13000	2	1.14125	1	1.15250	1
1.11875	1	1.13000	2	1.14125	1	1.15250	1	1.16375	1
1.13000	2	1.14125	1	1.15250	1	1.16375	1	1.17500	1
1.14125	1	1.15250	1	1.16375	1	1.17500	1	1.18625	1
1.15250	1	1.16375	1	1.17500	1	1.18625	1	1.19750	1
1.16375	1	1.17500	1	1.18625	1	1.19750	1	1.20875	1
1.17500	1	1.18625	1	1.19750	1	1.20875	1	1.22000	2
1.18625	1	1.19750	1	1.20875	1	1.22000	2	1.23125	1
1.19750	1	1.20875	1	1.22000	2	1.23125	1	1.24250	1
1.20875	1	1.22000	2	1.23125	1	1.24250	1	1.25375	1
1.22000	2	1.23125	1	1.24250	1	1.25375	1	1.26500	1
1.23125	1	1.24250	1	1.25375	1	1.26500	1	1.27625	1
1.24250	1	1.25375	1	1.26500	1	1.27625	1	1.28750	1
1.25375	1	1.26500	1	1.27625	1	1.28750	1	1.29875	1
1.26500	1	1.27625	1	1.28750	1	1.29875	1	1.31000	2
1.27625	1	1.28750	1	1.29875	1	1.31000	2	1.32125	1
1.28750	1	1.29875	1	1.31000	2	1.32125	1	1.33250	1
1.29875	1	1.31000	2	1.32125	1	1.33250	1	1.34375	1
1.31000	2	1.32125	1	1.33250	1	1.34375	1	1.35500	1
1.32125	1	1.33250	1	1.34375	1	1.35500	1	1.36625	1
1.33250	1	1.34375	1	1.35500	1	1.36625	1	1.37750	1
1.34375	1	1.35500	1	1.36625	1	1.37750	1	1.38875	1
1.35500	1	1.36625	1	1.37750	1	1.38875	1	1.40000	2
1.36625	1	1.37750	1	1.38875	1	1.40000	2	1.41125	1
1.37750	1	1.38875	1	1.40000	2	1.41125	1	1.42250	1
1.38875	1	1.40000	2	1.41125	1	1.42250	1	1.43375	1
1.40000	2	1.41125	1	1.42250	1	1.43375	1	1.44500	1
1.41125	1	1.42250	1	1.43375	1	1.44500	1	1.45625	1
1.42250	1	1.43375	1	1.44500	1	1.45625	1	1.46750	1
1.43375	1	1.44500	1	1.45625	1	1.46750	1	1.47875	1
1.44500	1	1.45625	1	1.46750	1	1.47875	1	1.49000	2
1.45625	1	1.46750	1	1.47875	1	1.49000	2	1.50125	1
1.46750	1	1.47875	1	1.49000	2	1.50125	1	1.51250	1
1.47875	1	1.49000	2	1.50125	1	1.51250	1	1.52375	1
1.49000	2	1.50125	1	1.51250	1	1.52375	1	1.53500	1
1.50125	1	1.51250	1	1.52375	1	1.53500	1	1.54625	1
1.51250	1	1.523							

TABLE 1.5-1 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_D , \bar{T} AND \bar{W}

$\bar{T}_D = -0.12000000 \text{ OZ}$				$\bar{T}_D = -0.12000000 \text{ OZ}$			
\bar{T}	\bar{y}	\bar{M}	$\bar{\lambda}$	\bar{T}	\bar{y}	\bar{M}	$\bar{\lambda}$
COMPRESSIVE END LOADS				COMPRESSIVE END LOADS			
0.48000000 OZ	0.60000000 OZ	0.	0.	0.5800571E 02	0.66250000 OZ	0.15000000 OZ	0.
0.5166988E 02	0.6235547E 01	0.5610192E 00	1.30000000 -00	0.6253272E 02	0.6882289E 01	0.2116000E 01	-0.30000000 -90
0.5682365E 02	0.6427832E 01	0.1020455E 01	0.40000000 -00	0.6635479E 02	0.7096370E 01	0.2635411E 01	-0.40000000 -00
0.5932400E 02	0.6695736E 01	0.1673927E 01	-0.50000000 -00	0.7180181E 02	0.7591171E 01	0.3347854E 01	-0.50000000 -00
0.6502303E 02	0.7054277E 01	0.2535900E 01	-0.60000000 -00	0.7944134E 02	0.7786115E 01	0.4303109E 01	-0.60000000 -00
0.7445922E 02	0.7529614E 01	0.3689511E 01	-0.70000000 -00	0.9015558E 02	0.8310039E 01	0.5571915E 01	-0.70000000 -00
0.8708125E 02	0.8162329E 01	0.5275890E 01	-0.80000000 -00	0.1034579E 03	0.9006680E 01	0.7264475E 01	-0.80000000 -00
0.1057193E 03	0.9018160E 01	0.7304712E 01	-0.90000000 -00	0.1280561E 03	0.9949691E 01	0.9559250E 01	-0.90000000 -00
0.1346707E 03	0.1029797E 02	0.1020979E 02	-0.10000000 01	0.1631625E 03	0.1126224E 02	0.1276274E 02	-0.10000000 01
0.1831254E 03	0.1196409E 02	0.1445525E 02	-0.11000000 01	0.2219236E 03	0.1317510E 02	0.1744187E 02	-0.11000000 01
0.2738652E 03	0.1466820E 02	0.2111644E 02	-0.12000000 01	0.3197171E 03	0.1616860E 02	0.2478249E 02	-0.12000000 01
0.4775244E 03	0.1944370E 02	0.3286001E 02	-0.13000000 01	0.4791066E 03	0.2143252E 02	0.3772096E 02	-0.13000000 01
0.1119784E 04	0.2989892E 02	0.5860188E 02	-0.14000000 01	0.1357968E 04	0.3294725E 02	0.6667661E 02	-0.14000000 01
0.2164108E 04	0.4165449E 02	0.8758273E 02	-0.15000000 01	0.2624030E 04	0.4599626E 02	0.9799680E 02	-0.15000000 01
0.6094746E 04	0.7006319E 02	0.1576420E 03	-0.15000000 01	0.7392127E 04	0.7718123E 02	0.1751510E 03	-0.15000000 01
0.6817483E 05	0.2351994E 03	0.5650665E 03	-0.15500000 01	0.8791214E 05	0.2590495E 03	0.6738863E 03	-0.15500000 01
TENSILE END LOADS				TENSILE END LOADS			
0.4470907E 02	0.5782945E 01	-0.5206650E 00	0.10000000 -02	0.5409852E 02	0.6585875E 01	0.9252712E 00	0.10000000 -06
0.4237845E 02	0.5624441E 01	-0.3999105E 00	0.40000000 -00	0.5127466E 02	0.6211252E 01	0.5061997E 00	0.40000000 -06
0.3964086E 02	0.5432693E 01	-0.1358173E 01	0.50000000 03	0.4795831E 02	0.6000000E 01	0.	0.50000000 00
0.3684097E 02	0.5214977E 01	-0.1877192E 01	0.60000000 02	0.44452475E 02	0.5760115E 01	-0.5736474E 00	0.10000000 00
0.3351485E 02	0.4978647E 01	-0.2439514E 01	0.70000000 00	0.4053097E 02	0.5499743E 01	-0.1194974E 01	0.10000000 00
0.3037980E 02	0.4710626E 01	-0.3027603E 01	0.80000000 06	0.3678124E 02	0.5228475E 01	-0.1844245E 01	0.10000000 00
0.2732892E 02	0.4477116E 01	0.3628464E 01	0.90000000 00	0.3305049E 02	0.4947142E 01	-0.2507135E 01	0.10000000 00
0.2447998E 02	0.4223349E 01	0.4223349E 01	0.10000000 01	0.2954485E 02	0.4667512E 01	-0.3167512E 01	0.10000000 01
0.2172710E 02	0.3973547E 01	0.4807991E 01	0.11000000 01	0.2627191E 02	0.4392215E 01	-0.3814605E 01	0.11000000 01
0.1924481E 02	0.3710949E 01	-0.5372566E 01	0.12000000 01	0.2327331E 02	0.4125881E 01	-0.4494982E 01	0.12000000 01
0.1697112E 02	0.3497902E 01	0.5911553E 01	0.13000000 01	0.2055009E 02	0.3868015E 01	-0.5036779E 01	0.13000000 01
0.1476184E 02	0.3275988E 01	0.6420936E 01	0.14000000 01	0.1810184E 02	0.3621438E 01	-0.5601919E 01	0.14000000 01
0.1315227E 02	0.3066154E 01	-0.6898480E 01	0.15000000 01	0.1597918E 02	0.3392131E 01	-0.6131102E 01	0.15000000 01
0.1154078E 02	0.2888852E 01	0.7344261E 01	0.16000000 01	0.1397831E 02	0.3174620E 01	-0.6627388E 01	0.16000000 01
0.1011211E 02	0.2674943E 01	-0.7757192E 01	0.17000000 01	0.1225977E 02	0.2970948E 01	-0.7086154E 01	0.17000000 01
0.8847665E 01	0.2511136E 01	-0.8136541E 01	0.18000000 01	0.1076407E 02	0.2783392E 01	-0.7510451E 01	0.18000000 01
0.7728652E 01	0.2351366E 01	-0.8488099E 01	0.19000000 01	0.9199204E 01	0.2608145E 01	-0.7901075E 01	0.19000000 01
0.6737868E 01	0.2202493E 01	0.8810571E 01	0.20000000 01	0.8213378E 01	0.2454931E 01	-0.8252775E 01	0.20000000 01
0.5959216E 01	0.2064504E 01	-0.9108464E 01	0.21000000 01	0.7165470E 01	0.2287600E 01	-0.8581403E 01	0.21000000 01
0.5078207E 01	0.1936565E 01	-0.9372977E 01	0.22000000 01	0.6216061E 01	0.2138454E 01	-0.8888415E 01	0.22000000 01
0.4341759E 01	0.1818047E 01	-0.9617734E 01	0.23000000 01	0.5629988E 01	0.201110E 01	-0.9181209E 01	0.23000000 01
0.3758709E 01	0.1702827E 01	-0.9880541E 01	0.24000000 01	0.5137319E 01	0.1894441E 01	-0.9461144E 01	0.24000000 01
0.3198110E 01	0.1608993E 01	-0.1009134E 02	0.25000000 01	0.4013851E 01	0.1782627E 01	-0.9681619E 01	0.25000000 01
0.2694744E 01	0.1517901E 01	0.1022727E 02	0.26000000 01	0.3420951E 01	0.1678788E 01	-0.9846208E 01	0.26000000 01
0.2233273E 01	0.1425813E 01	0.1039412E 02	0.27000000 01	0.2885181E 01	0.1582697E 01	-0.1011766E 02	0.27000000 01
0.1818208E 01	0.1355145E 01	-0.1055593E 02	0.28000000 01	0.2399160E 01	0.1493576E 01	-0.1224455E 02	0.28000000 01
0.1429609E 01	0.1270124E 01	-0.106614E 02	0.29000000 01	0.1955181E 01	0.1410921E 01	-0.1246545E 02	0.29000000 01
0.1074551E 01	0.1200896E 01	-0.1080006E 02	0.30000000 01	0.1540100E 01	0.1334708E 01	-0.1050784E 02	0.30000000 01
COMPRESSIVE END LOADS				COMPRESSIVE END LOADS			
0.6018256E 02	0.7750000E 01	0.5000000E 01	0.	0.6171716E 02	0.7875000E 01	0.4500000E 01	0.
0.7444111E 02	0.7510111E 01	0.1677773E 01	0.10000000 00	0.8159660E 02	0.8197773E 01	0.5216180E 01	0.10000000 00
0.7889675E 02	0.7764808E 01	0.4245691E 01	0.40000000 00	0.8278957E 02	0.8415296E 01	0.5862727E 01	0.40000000 00
0.8549759E 02	0.8087172E 01	0.5021781E 01	0.50000000 00	0.1203078E 03	0.8895708E 01	0.6500000E 01	0.50000000 00
0.9459692E 02	0.8518550E 01	0.6066678E 01	0.60000000 00	0.1119811E 03	0.9250687E 01	0.7383277E 01	0.60000000 00
0.1071644E 03	0.9090740E 01	0.7455318E 01	-0.70000000 00	0.1281111E 03	0.9870841E 01	0.8336722E 01	-0.70000000 00
0.1256054E 03	0.9651640E 01	0.9105055E 01	0.80000000 00	0.1475215E 03	0.1067611E 02	0.9335881E 01	0.80000000 00
0.1525457E 03	0.1066112E 02	0.1181379E 02	0.90000000 00	0.1791861E 03	0.1181275E 02	0.1050613E 02	0.90000000 00
0.1948066E 03	0.1213686E 02	0.1531408E 02	0.10000000 01	0.2283851E 03	0.1336731E 02	0.1366731E 02	0.10000000 01
0.2688620E 03	0.1440372E 02	0.2042850E 02	0.11000000 01	0.3107580E 03	0.1654253E 02	0.2435112E 02	0.11000000 01
0.3954801E 03	0.1767250E 02	0.2944658E 02	0.12000000 01	0.4084908E 03	0.1917664E 02	0.3111459E 02	0.12000000 01
0.4901615E 03	0.2147173E 02	0.4456819E 02	0.13000000 01	0.5119059E 03	0.2258227E 02	0.3946275E 02	0.13000000 01
0.6191427E 03	0.2609792E 02	0.7155134E 02	0.14000000 01	0.6302429E 03	0.2702493E 02	0.5004493E 02	0.14000000 01
0.8129562E 03	0.3136131E 02	0.1064133E 03	0.15000000 01	0.8670272E 03	0.3417507E 02	0.6711457E 02	0.15000000 01
0.1118627E 04	0.4622997E 02	0.1926715E 03	0.16000000 01	0.1136213E 04	0.4714775E 02	0.9114775E 02	0.16000000 01
0.1382331E 04	0.2828946E 03	0.4546662E 03	0.17000000 01	0.1332505E 04	0.3967627E 03	0.7454714E 03	0.17000000 01
TENSILE END LOADS				TENSILE END LOADS			
0.6452331E 02	0.6537966E 01	0.2371005E 01	0.10000000 00	0.7552285E 02	0.7721230E 01	0.3307388E 01	0.10000000 00
0.6802754E 02	0.6799663E 01	0.1912310E 01	0.40000000 00	0.7711675E 02	0.7944354E 01	0.3487184E 01	0.40000000 00
0.7276767E 02	0.7058117E 01	0.1354933E 01	0.50000000 00	0.8087150E 02	0.8284441E 01	0.3712667E 01	0.50000000 00
0.7794819E 02	0.7305981E 01	0.7300000E 00	0.60000000 00	0.8473150E 02	0.8614954E 01	0.3946667E 01	0.60000000 00
0.8371913E 02	0.7567867E 01	0.2786873E 01	0.70000000 00	0.8972350E 02	0.9014954E 01	0.4194667E 01	0.70000000 00
0.9037971E 02	0.7822336E 01	0.6822717E 01	0.80000000 00	0.9591210E 02	0.9481774E 01	0.4462777E 01	0.80000000 00
0.1042271E 03	0.8281716E 01	0.1387228E 01	0.90000000 00	0.1042854E 03	0.9955557E 01	0.4755777E 01	0.90000000 00
0.1151877E 03	0.8711674E 01	0.2111674E 01	0.10000000 01	0.1142854E 03	0.1042854E 01	0.5072777E 01	0.10000000 01
0.1275814E 03	0.915107E 01	0.2827129E 01	0.11000000 01	0.1242854E 03	0.1092774E 01	0.5402777E 01	0.11000000 01
0.1409904E 03	0.96510E 01	0.3507072E 01	0.12000000 01	0.13			

TABLE 1.5-1 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_D , \bar{T} AND \bar{W}

\bar{T}_D	\bar{y}	\bar{M}	$\bar{\lambda}$	\bar{T}_D	\bar{y}	\bar{M}	$\bar{\lambda}$
-0.1200000E 02				-0.1200000E 02			
COMPRESSIVE END LOADS				COMPRESSIVE END LOADS			
0.941714E 02	0.8500000E 01	0.6000000E 01	0.	0.1091429E 03	0.9125000E 01	0.7500000E 01	0.
0.1013981E 03	0.8428514E 01	0.6794566E 01	-0.1000000E 00	0.1168460E 03	0.9477257E 01	0.8152953E 01	-0.1000000E 00
0.1076131E 03	0.9101784E 01	0.7456285E 01	-0.4000000E 00	0.1158775E 03	0.9777077E 01	0.9363293E 01	-0.4000000E 00
0.1168814E 03	0.9478542E 01	0.8169636E 01	-0.5000000E 00	0.1137778E 03	0.1017825E 02	0.1004358E 02	-0.5000000E 00
0.1288973E 03	0.9982823E 01	0.952116E 01	-0.5000000E 00	0.1168460E 03	0.1074996E 02	0.1115758E 02	-0.5000000E 00
0.1463193E 03	0.1065179E 02	0.1121174E 02	-0.7000000E 00	0.1168894E 03	0.1141689E 02	0.1110153E 02	-0.7000000E 00
0.1712132E 03	0.1156297E 02	0.1338622E 02	-0.7000000E 00	0.11966735E 03	0.1235865E 02	0.1154268E 02	-0.7000000E 00
0.2079833E 03	0.1274498E 02	0.1632277E 02	-0.7000000E 00	0.1269312E 03	0.1367591E 02	0.1457741E 02	-0.7000000E 00
0.2651158E 03	0.1441454E 02	0.1774195E 02	-0.7000000E 00	0.1345529E 03	0.1547202E 02	0.1727202E 02	-0.7000000E 00
0.3607594E 03	0.1686094E 02	0.1860174E 02	-0.7000000E 00	0.14155184E 03	0.1808750E 02	0.2038816E 02	-0.7000000E 00
0.5199037E 03	0.2068100E 02	0.1578154E 02	-0.7000000E 00	0.16204188E 03	0.2218520E 02	0.2494669E 02	-0.7000000E 00
0.7422360E 03	0.2739870E 02	0.1524180E 02	-0.7000000E 00	0.1837851E 04	0.2938782E 02	0.3716675E 02	-0.7000000E 00
0.1010122E 04	0.4092244E 02	0.8953079E 02	-0.7000000E 00	0.2550384E 04	0.4514057E 02	0.7597552E 02	-0.7000000E 00
0.4272453E 04	0.5881588E 02	0.1292732E 03	-0.7000000E 00	0.4910613E 04	0.8285535E 02	0.1396534E 03	-0.7000000E 00
0.1201649E 05	0.7851563E 02	0.2277051E 03	-0.7000000E 00	0.1581277E 05	0.1056532E 03	0.2451109E 03	-0.7000000E 00
0.1502722E 06	0.1010122E 03	0.4002553E 03	-0.7000000E 00	0.1551977E 06	0.1554498E 03	0.4592657E 03	-0.7000000E 00
TENSILE END LOADS				TENSILE END LOADS			
0.8769772E 02	0.8124666E 01	0.5782491E 01	0.1000000E 00	0.1010122E 03	0.9777077E 01	0.9777077E 01	0.1000000E 00
0.0510132E 02	0.7771686E 01	0.4724533E 01	0.4000000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.4000000E 00
0.7771686E 02	0.7701723E 01	0.4074520E 01	0.5000000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.5000000E 00
0.7771686E 02	0.7701723E 01	0.4074520E 01	0.6000000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.6000000E 00
0.6566675E 02	0.7061966E 01	0.2551917E 01	0.7000000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.7000000E 00
0.5950079E 02	0.6719032E 01	0.1733328E 01	0.8000000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.8000000E 00
0.5151909E 02	0.6531775E 01	0.1458657E 01	0.9000000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.9000000E 00
0.4781131E 02	0.6303064E 01	0.	0.1000000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.1000000E 00
0.4510175E 02	0.6080131E 01	0.0384447E 01	0.1100000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.1100000E 00
0.7683201E 02	0.5306674E 01	0.1681617E 01	0.1200000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.1200000E 00
0.5128169E 02	0.4973841E 01	0.2613553E 01	0.1300000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.1300000E 00
0.2913659E 02	0.4665790E 01	0.3184948E 01	0.1400000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.1400000E 00
0.2581505E 02	0.4370066E 01	0.4126893E 01	0.1500000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.1500000E 00
0.2207881E 02	0.4071957E 01	0.4475409E 01	0.1600000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.1600000E 00
0.1991173E 02	0.3813104E 01	0.5073042E 01	0.1700000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.1700000E 00
0.1750999E 02	0.3588828E 01	0.5676508E 01	0.1800000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.1800000E 00
0.1537229E 02	0.3387162E 01	0.6147481E 01	0.1900000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.1900000E 00
0.134910E 02	0.3197455E 01	0.6607700E 01	0.2000000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.2000000E 00
0.1181922E 02	0.2954071E 01	0.7010661E 01	0.2100000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.2100000E 00
0.1017667E 02	0.2724146E 01	0.7416811E 01	0.2200000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.2200000E 00
0.9081494E 02	0.2508627E 01	0.7799637E 01	0.2300000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.2300000E 00
0.7942663E 02	0.2314962E 01	0.8132113E 01	0.2400000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.2400000E 00
0.6927911E 02	0.2109799E 01	0.8416242E 01	0.2500000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.2500000E 00
0.6021756E 02	0.2176875E 01	0.8714312E 01	0.2600000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.2600000E 00
0.5216946E 02	0.2051290E 01	0.8960887E 01	0.2700000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.2700000E 00
0.4488460E 02	0.1938876E 01	0.9200790E 01	0.2800000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.2800000E 00
0.3835049E 02	0.1832712E 01	0.9431304E 01	0.2900000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.2900000E 00
0.3239011E 02	0.1734150E 01	0.9607169E 01	0.3000000E 00	0.0742376E 03	0.4554476E 01	0.6150643E 01	0.3000000E 00
TENSILE END LOADS				TENSILE END LOADS			
0.1210857E 03	0.7755300E 01	0.9000000E 01	0.	0.1210857E 03	0.7755300E 01	0.9000000E 01	0.
0.1325004E 03	0.1012600E 02	0.8911500E 01	0.1000000E 00	0.1325004E 03	0.1012600E 02	0.8911500E 01	0.1000000E 00
0.1406727E 03	0.1061676E 02	0.1061676E 02	0.2000000E 00	0.1406727E 03	0.1061676E 02	0.1061676E 02	0.2000000E 00
0.1522710E 03	0.1088790E 02	0.1117179E 02	0.3000000E 00	0.1522710E 03	0.1088790E 02	0.1117179E 02	0.3000000E 00
0.1652242E 03	0.1148710E 02	0.1132005E 02	0.4000000E 00	0.1652242E 03	0.1148710E 02	0.1132005E 02	0.4000000E 00
0.1913239E 03	0.1271211E 02	0.1492919E 02	0.5000000E 00	0.1913239E 03	0.1271211E 02	0.1492919E 02	0.5000000E 00
0.2130046E 03	0.1342072E 02	0.1746718E 02	0.6000000E 00	0.2130046E 03	0.1342072E 02	0.1746718E 02	0.6000000E 00
0.2715201E 03	0.1456174E 02	0.2003193E 02	0.7000000E 00	0.2715201E 03	0.1456174E 02	0.2003193E 02	0.7000000E 00
0.3461484E 03	0.1651472E 02	0.2552447E 02	0.8000000E 00	0.3461484E 03	0.1651472E 02	0.2552447E 02	0.8000000E 00
0.4720176E 03	0.1811817E 02	0.4237497E 02	0.9000000E 00	0.4720176E 03	0.1811817E 02	0.4237497E 02	0.9000000E 00
0.7661360E 03	0.2568490E 02	0.8113172E 02	0.1000000E 01	0.7661360E 03	0.2568490E 02	0.8113172E 02	0.1000000E 01
0.1112300E 04	0.4117451E 02	0.4702770E 02	0.1100000E 01	0.1112300E 04	0.4117451E 02	0.4702770E 02	0.1100000E 01
0.2014088E 04	0.6161649E 02	0.1016502E 03	0.1200000E 01	0.2014088E 04	0.6161649E 02	0.1016502E 03	0.1200000E 01
0.3491482E 04	0.8729512E 02	0.1500675E 03	0.1300000E 01	0.3491482E 04	0.8729512E 02	0.1500675E 03	0.1300000E 01
0.5755572E 04	0.1127718E 03	0.2827167E 03	0.1400000E 01	0.5755572E 04	0.1127718E 03	0.2827167E 03	0.1400000E 01
0.1077744E 06	0.1782902E 03	0.9178850E 03	0.1500000E 01	0.1077744E 06	0.1782902E 03	0.9178850E 03	0.1500000E 01
TENSILE END LOADS				TENSILE END LOADS			
0.1146210E 03	0.9005291E 01	0.3154957E 01	0.1000000E 00	0.1254310E 03	0.1000344E 02	0.9594689E 01	0.1000000E 00
0.1088057E 03	0.9185108E 01	0.3751671E 01	0.2000000E 00	0.1226227E 03	0.9732119E 01	0.9732119E 01	0.2000000E 00
0.1015593E 03	0.8816553E 01	0.4703687E 01	0.3000000E 00	0.1168522E 03	0.9818199E 01	0.8149031E 01	0.3000000E 00
0.9181150E 02	0.3485007E 01	0.7953784E 01	0.4000000E 00	0.1339641E 03	0.9031082E 01	0.7798818E 01	0.4000000E 00
0.8580170E 02	0.6105298E 01	0.9028630E 01	0.5000000E 00	0.9688464E 02	0.8826150E 01	0.6271086E 01	0.5000000E 00
0.7738492E 02	0.7705721E 01	0.4086339E 01	0.6000000E 00	0.6778715E 02	0.4701570E 01	0.5250895E 01	0.6000000E 00
0.6911802E 02	0.7297271E 01	0.1086219E 01	0.7000000E 00	0.7594715E 02	0.4203489E 01	0.4203489E 01	0.7000000E 00
0.6288641E 02	0.6888126E 01	0.2111674E 01	0.8000000E 00	0.7955155E 02	0.7332688E 01	0.5167531E 01	0.8000000E 00
0.5556895E 02	0.6456791E 01	0.3152329E 01	0.9000000E 00	0.8274015E 02	0.6924368E 01	0.2145715E 01	0.9000000E 00
0.4972709E 02	0.6094581E 01	0.2218571E 00</					

TABLE 1.5-1 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_D , \bar{T} AND \bar{W}

\bar{T}_D	\bar{y}	\bar{M}	$\bar{\lambda}$	\bar{T}_D	\bar{y}	\bar{M}	$\bar{\lambda}$
-0.1200000E 02				-0.1600000E 02			
COMPRESSIVE END LOADS				COMPRESSIVE END LOADS			
0.1558857E 03	0.1100000E 02	0.1200000E 02	0.	0.8533333E 02	0.8000000E 01	0.	0.
0.1678674E 03	0.1152340E 02	0.1102811E 02	-0.1000000E -03	0.9184519E 02	0.8311396E 01	0.7480757E 00	-0.3000000E -00
0.1781756E 03	0.1171730E 02	0.1086412E 02	-0.4000000E -00	0.9744352E 02	0.8570441E 01	0.1371271E 01	-0.4000000E -00
0.1920837E 03	0.1226118E 02	0.1050634E 02	-0.5000000E -00	0.1034325E 03	0.8927611E 01	0.2231903E 01	-0.5000000E -00
0.2134776E 03	0.1291137E 02	0.1064809E 02	-0.6000000E -00	0.1166165E 03	0.9405102E 01	0.3586053E 01	-0.6000000E -00
0.2423784E 03	0.1377294E 02	0.1074374E 02	-0.7000000E -00	0.1323084E 03	0.1003949E 02	0.4919348E 01	-0.7000000E -00
0.2836796E 03	0.1491726E 02	0.2154855E 02	-0.8000000E -00	0.1547201E 03	0.1088310E 02	0.6965187E 01	-0.8000000E -00
0.3446918E 03	0.1647341E 02	0.2534103E 02	-0.9000000E -00	0.1878403E 03	0.1262421E 02	0.9739613E 01	-0.9000000E -00
0.3846918E 03	0.1862757E 02	0.3062757E 02	-0.1200000E 01	0.2392845E 03	0.1361305E 02	0.1361305E 02	-0.1000000E 01
0.4395021E 03	0.2177540E 02	0.3834023E 02	-0.1100000E 01	0.3253995E 03	0.1592965E 02	0.1927367E 02	-0.1100000E 01
0.5982366E 03	0.2777540E 02	0.5046483E 02	-0.1200000E 01	0.4466848E 03	0.1955226E 02	0.2815526E 02	-0.1200000E 01
0.8955770E 03	0.2669780E 02	0.7048483E 02	-0.1200000E 01	0.6468909E 03	0.2592506E 02	0.4181335E 02	-0.1300000E 01
0.1563402E 04	0.3515160E 02	0.7747592E 02	-0.1300000E 01	0.1999472E 04	0.3986522E 02	0.7813584E 02	-0.1400000E 01
0.3688361E 04	0.5428557E 02	0.1151177E 03	-0.1400000E 01	0.1847630E 04	0.5554198E 02	0.1167770E 03	-0.1456000E 01
0.7091740E 04	0.7547667E 02	0.1708957E 03	-0.1500000E 01	0.1033481E 05	0.9341746E 02	0.2101893E 03	-0.1500000E 01
0.1997694E 05	0.1270001E 03	0.2977807E 03	-0.1500000E 01	0.1215549E 06	0.1115797E 03	0.7534220E 03	-0.1550000E 01
0.2241410E 06	0.4280001E 03	0.1035465E 04	-0.1550000E 01				
TENSILE END LOADS				TENSILE END LOADS			
0.1451682E 03	0.1060639E 02	0.1104542E 02	0.3000000E -00	0.7749444E 02	0.7710753E 01	-0.6932534E 00	0.3000000E -00
0.1375351E 03	0.1011493E 02	0.1034897E 02	0.4000000E -00	0.7536021E 02	0.7499255E 01	0.1199481E 01	0.4000000E -00
0.1286060E 03	0.9971140E 01	0.9507214E 01	0.5000000E -00	0.7050505E 02	0.7243591E 01	-0.1810498E 01	0.5000000E -00
0.1188268E 03	0.9576216E 01	0.8552562E 01	0.6000000E -00	0.6516816E 02	0.6953102E 01	-0.2501389E 01	0.6000000E -00
0.1086424E 03	0.9147451E 01	0.7517749E 01	0.7000000E -00	0.5964547E 02	0.6638189E 01	-0.3252112E 01	0.7000000E -00
0.9843814E 02	0.8697419E 01	0.6433652E 01	0.8000000E -00	0.5429169E 02	0.6107502E 01	-0.4035460E 01	0.8000000E -00
0.8851964E 02	0.8217121E 01	0.5127759E 01	0.9000000E -00	0.4848978E 02	0.5596949E 01	-0.4835266E 01	0.9000000E -00
0.7910820E 02	0.7776651E 01	0.4221349E 01	0.1000000E 01	0.4358067E 02	0.5157112E 01	-0.5657152E 01	0.1000000E 01
0.7034075E 02	0.7323056E 01	0.3139102E 01	0.1100000E 01	0.3787390E 02	0.4799302E 01	-0.6431655E 01	0.1100000E 01
0.6232077E 02	0.6882409E 01	0.2089113E 01	0.1200000E 01	0.3439230E 02	0.4497449E 01	-0.7181421E 01	0.1200000E 01
0.5505225E 02	0.6458956E 01	0.1084350E 01	0.1300000E 01	0.3067552E 02	0.4064972E 01	-0.7811920E 01	0.1300000E 01
0.4853402E 02	0.6055592E 01	0.1310126E -00	0.1400000E 01	0.2685644E 02	0.3614728E 01	-0.8561748E 01	0.1400000E 01
0.4273133E 02	0.5674017E 01	-0.7665106E 00	0.1500000E 01	0.2327139E 02	0.3208973E 01	-0.9194463E 01	0.1500000E 01
0.3759391E 02	0.5315061E 01	-0.1600557E 01	0.1600000E 01	0.2004912E 02	0.2825136E 01	-0.9727234E 01	0.1600000E 01
0.3306339E 02	0.4978856E 01	-0.2104893E 01	0.1700000E 01	0.1645208E 02	0.2457866E 01	-0.1034292E 02	0.1700000E 01
0.2907862E 02	0.4665021E 01	-0.3136747E 01	0.1800000E 01	0.1316919E 02	0.2169112E 01	-0.1085112E 02	0.1800000E 01
0.2557027E 02	0.4372828E 01	-0.4189928E 01	0.1900000E 01	0.1025773E 02	0.1835427E 01	-0.1174718E 02	0.1900000E 01
0.2250827E 02	0.4107294E 01	-0.5405101E 01	0.2000000E 01	0.7749444E 02	0.1527679E 01	-0.1174718E 02	0.2000000E 01
0.1981259E 02	0.3869110E 01	-0.6475661E 01	0.2100000E 01	0.7050505E 02	0.1271597E 01	-0.1215977E 02	0.2100000E 01
0.1764457E 02	0.3615617E 01	-0.7439986E 01	0.2200000E 01	0.6516816E 02	0.1042649E 01	-0.1242750E 02	0.2200000E 01
0.1516150E 02	0.3399157E 01	-0.8458151E 01	0.2300000E 01	0.6067062E 02	0.0826213E 01	-0.1262354E 02	0.2300000E 01
0.1352566E 02	0.3198557E 01	-0.9421686E 01	0.2400000E 01	0.5682831E 02	0.0627793E 01	-0.1277933E 02	0.2400000E 01
0.1190399E 02	0.3012694E 01	-0.6829319E 01	0.2500000E 01	0.5362831E 02	0.0431254E 01	-0.1285206E 02	0.2500000E 01
0.1066769E 02	0.2840454E 01	0.7201613E 01	0.2600000E 01	0.5062831E 02	0.0271720E 01	-0.1285206E 02	0.2600000E 01
0.9191761E 01	0.2680748E 01	-0.7542654E 01	0.2700000E 01	0.4781527E 02	0.1901131E 01	-0.1285206E 02	0.2700000E 01
0.8055561E 01	0.2532600E 01	-0.7855656E 01	0.2800000E 01	0.4521555E 02	0.1591526E 01	-0.1285206E 02	0.2800000E 01
0.7037408E 01	0.2395099E 01	-0.8142780E 01	0.2900000E 01	0.4281717E 02	0.1291576E 01	-0.1285206E 02	0.2900000E 01
0.6124175E 01	0.2267164E 01	-0.8406275E 01	0.3000000E 01	0.4057879E 02	0.1001139E 01	-0.1285206E 02	0.3000000E 01
TENSILE END LOADS				TENSILE END LOADS			
0.9661901E 02	0.8475003E 01	0.1500302E 01	0.	0.1126752E 03	0.7757712E 01	0.1473322E 01	0.
0.1361884E 03	0.9460131E 01	0.2568831E 01	0.1000000E -00	0.1215112E 03	0.8426673E 01	0.3386472E 01	0.1000000E -00
0.1126332E 03	0.9219431E 01	0.2476124E 01	0.1000000E -00	0.1269435E 03	0.8749733E 01	0.4495137E 01	0.1000000E -00
0.1126760E 03	0.9021552E 01	0.2405051E 01	0.1000000E -00	0.1339435E 03	0.9134933E 01	0.5757737E 01	0.1000000E -00
0.1190273E 03	0.8813784E 01	0.2318422E 01	0.1000000E -00	0.1436435E 03	0.9596249E 01	0.6991713E 01	0.1000000E -00
0.1322480E 03	0.8580179E 01	0.2207175E 01	0.1000000E -00	0.1553737E 03	0.1013613E 02	0.8084553E 01	0.1000000E -00
0.1489352E 03	0.8312777E 01	0.2082769E 01	0.1000000E -00	0.1692431E 03	0.1057242E 02	0.9135853E 01	0.1000000E -00
0.1712700E 03	0.7924578E 01	0.1912943E 01	0.1000000E -00	0.1856944E 03	0.1119977E 02	0.1024348E 02	0.1000000E -00
0.2078016E 03	0.7404552E 01	0.1616551E 01	0.1000000E -00	0.2053139E 03	0.1207375E 02	0.1167175E 02	0.1000000E -00
0.2605070E 03	0.6718277E 01	0.1228329E 01	0.1000000E -00	0.2283549E 03	0.1318059E 02	0.1292459E 02	0.1000000E -00
0.3363225E 03	0.5823233E 01	0.0812133E 01	0.1000000E -00	0.2543549E 03	0.1456858E 02	0.1504673E 02	0.1000000E -00
0.4426538E 03	0.4733139E 01	0.0495742E 01	0.1000000E -00	0.3222137E 04	0.2099225E 02	0.2454525E 02	0.1000000E -00
0.5921827E 03	0.3423135E 01	0.0255158E 01	0.1000000E -00	0.4255742E 04	0.2878186E 02	0.3414529E 02	0.1000000E -00
0.7885364E 03	0.2027131E 01	0.0127131E 01	0.1000000E -00	0.5686354E 04	0.3832153E 02	0.4632153E 02	0.1000000E -00
0.1074176E 04	0.0554433E 01	0.0022219E 01	0.1000000E -00	0.7535734E 04	0.5122494E 02	0.6177137E 02	0.1000000E -00
TENSILE END LOADS				TENSILE END LOADS			
0.9108270E 02	0.8071524E 01	0.1500302E 01	0.	0.1205354E 03	0.8071524E 01	0.1500302E 01	0.
0.8022740E 02	0.7626262E 01	0.1276727E 01	0.1000000E -00	0.1276727E 03	0.7626262E 01	0.1276727E 01	0.1000000E -00
0.7018635E 02	0.7012899E 01	0.1045276E 01	0.1000000E -00	0.1345139E 03	0.6912899E 01	0.1045276E 01	0.1000000E -00
0.6130740E 02	0.6242987E 01	0.0819948E 01	0.1000000E -00	0.1412899E 03	0.6130740E 01	0.0819948E 01	0.1000000E -00
0.5360975E 02	0.5360975E 01	0.0536097E 01	0.1000000E -00	0.1480975E 03	0.5360975E 01	0.0536097E 01	0.1000000E -00
0.4682674E 02	0.4682674E 01	0.0468267E 01	0.1000000E -00	0.1548267E 03	0.4682674E 01	0.0468267E 01	0.1000000E -00
0.4092759E 02	0.4092759E 01	0.0409275E 01	0.1000000E -00	0.1615759E 03	0.4092759E 01	0.0409275E 01	0.1000000E -00
0.3587723E 02	0.3587723E 01	0.0358772E 01	0.1000000E -00	0.168323E 03	0.3587723E 01	0.0358772E 01	0.1000000E -00
0.3169780E 02	0.3169780E 01	0.0316978E 01	0.1000000E -00	0.1750780E 03	0.3169780E 01	0.0316978E 01	0.1000000E -00
0.2811492E 02	0.2811492E 01	0.0281149E 01	0.1000000E -00	0.1818492E 03	0.2811492E 01	0.0281149E 01	0.1000000

TABLE 1.5-1 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_D , \bar{T} AND \bar{W}

\bar{T}_D	-0.16000000 02	\bar{W}	0.70000000 01	\bar{T}_D	-0.16000000 02	\bar{W}	0.12000000 02
\bar{T}	\bar{y}	\bar{M}	$\bar{\lambda}$	\bar{T}	\bar{y}	\bar{M}	$\bar{\lambda}$
COMPRESSIVE END LOADS				COMPRESSIVE END LOADS			
0.1281048E 03	0.9875000E 01	0.4500000E 01	0.	0.1443048E 03	0.1050000E 02	0.6000000E 01	0.
0.1379121E 03	0.1025762E 02	0.5423186E 01	-0.3000000E -00	0.1553602E 03	0.1090000E 02	0.6981573E 01	-0.1000000E -00
0.1463451E 03	0.1057591E 02	0.6192145E 01	-0.4000000E -00	0.1648673E 03	0.1124439E 02	0.7799103E 01	-0.4000000E -00
0.1583801E 03	0.1101474E 02	0.7253688E 01	-0.5000000E -00	0.1784349E 03	0.1171044E 02	0.8927611E 01	-0.5000000E -00
0.1752301E 03	0.1160211E 02	0.8676760E 01	-0.6000000E -00	0.1974332E 03	0.1231425E 02	0.1044033E 02	-0.6000000E -00
0.1998749E 03	0.1238073E 02	0.1056656E 02	-0.7000000E -00	0.2240886E 03	0.1316115E 02	0.1244896E 02	-0.7000000E -00
0.2326620E 03	0.1341708E 02	0.1308693E 02	-0.8000000E -00	0.2621817E 03	0.1429174E 02	0.1512752E 02	-0.8000000E -00
0.2825696E 03	0.1481881E 02	0.1650432E 02	-0.9000000E -00	0.3184516E 03	0.1575034E 02	0.1875777E 02	-0.9000000E -00
0.3601179E 03	0.1677039E 02	0.2127019E 02	-0.1000000E 01	0.4058882E 03	0.1762794E 02	0.2392264E 02	-0.1000000E 01
0.4879427E 03	0.1961449E 02	0.2823554E 02	-0.1100000E 01	0.5122709E 03	0.2004111E 02	0.3122016E 02	-0.1100000E 01
0.631157E 03	0.2406486E 02	0.3915340E 02	-0.1200000E 01	0.6264637E 03	0.2256206E 02	0.4281245E 02	-0.1200000E 01
0.8129257E 04	0.3109124E 02	0.5839619E 02	-0.1300000E 01	0.7642282E 04	0.2518796E 02	0.6257144E 02	-0.1300000E 01
0.1000629E 04	0.4091022E 02	0.1005600E 03	-0.1400000E 01	0.9383284E 04	0.3205855E 02	0.1080330E 03	-0.1400000E 01
0.5800143E 04	0.6262110E 02	0.1480194E 03	-0.1450000E 01	0.1165999E 04	0.4225010E 02	0.1584335E 03	-0.1450000E 01
0.1635712E 05	0.1147718E 03	0.2627136E 03	-0.1500000E 01	0.1482214E 05	0.5721890E 02	0.2802524E 03	-0.1500000E 01
0.1832918E 06	0.1851494E 03	0.9298216E 03	-0.1550000E 01	0.2068812E 06	0.8089995E 02	0.4586214E 04	-0.1550000E 01
TENSILE END LOADS				TENSILE END LOADS			
0.1191515E 03	0.9519185E 01	0.1641255E 01	0.4000000E -00	0.1343989E 03	0.1012231E 02	0.5088922E 01	0.3000000E 00
0.1130867E 03	0.9259688E 01	0.1018450E 01	0.4000000E -00	0.1271756E 03	0.9864697E 01	0.4424560E 01	0.4000000E -00
0.1057772E 03	0.8995511E 01	0.2263622E 01	0.5000000E 00	0.1191364E 03	0.9512817E 01	0.3621776E 01	0.5000000E 00
0.9777091E 02	0.8588767E 01	0.1400344E 01	0.6000000E 00	0.1101122E 03	0.9131227E 01	0.2711788E 01	0.6000000E 00
0.8942555E 02	0.8201497E 01	0.4812687E -00	0.7000000E 00	0.1007140E 03	0.8722594E 01	0.1725429E 01	0.7000000E 00
0.8107662E 02	0.7795049E 01	-0.4888314E -00	0.8000000E 00	0.9129673E 02	0.8209089E 01	0.6938253E -00	0.8000000E 00
0.7295265E 02	0.7379568E 01	-0.1477448E 01	0.9000000E 00	0.8214158E 02	0.7849422E 01	-0.1581695E -00	0.9000000E 00
0.6524215E 02	0.6943250E 01	-0.2434201E 01	0.1000000E 01	0.7345525E 02	0.7407783E 01	-0.1407783E -01	0.1000000E 01
0.5806356E 02	0.6554120E 01	-0.1430495E 01	0.1100000E 01	0.6516528E 02	0.6972817E 01	-0.2431008E -01	0.1100000E 01
0.5148188E 02	0.6156196E 01	-0.4365210E 01	0.1200000E 01	0.5795094E 02	0.6450120E 01	-0.3432473E -01	0.1200000E 01
0.4552019E 02	0.5774268E 01	-0.5258513E 01	0.1300000E 01	0.5121625E 02	0.5944018E 01	-0.4380438E -01	0.1300000E 01
0.4017093E 02	0.5410355E 01	-0.6104257E 01	0.1400000E 01	0.4521267E 02	0.5457786E 01	-0.5285260E -01	0.1400000E 01
0.3540578E 02	0.5066154E 01	-0.6898848E 01	0.1500000E 01	0.3984825E 02	0.5027137E 01	-0.6132109E -01	0.1500000E 01
0.3110389E 02	0.4742465E 01	-0.7640710E 01	0.1600000E 01	0.3500671E 02	0.4621497E 01	-0.6923497E -01	0.1600000E 01
0.2745210E 02	0.4439181E 01	-0.8328810E 01	0.1700000E 01	0.3090425E 02	0.4226219E 01	-0.7587731E -01	0.1700000E 01
0.2417575E 02	0.4156550E 01	-0.8947246E 01	0.1800000E 01	0.2721456E 02	0.3842706E 01	-0.8192880E -01	0.1800000E 01
0.2129069E 02	0.3895327E 01	-0.9553710E 01	0.1900000E 01	0.2397238E 02	0.3444594E 01	-0.8687036E -01	0.1900000E 01
0.1875510E 02	0.3648805E 01	-0.1009422E 02	0.2000000E 01	0.2112411E 02	0.3088134E 01	-0.9444571E -01	0.2000000E 01
0.1652503E 02	0.3421975E 01	-0.1059091E 02	0.2100000E 01	0.1862700E 02	0.2640768E 01	-0.1007674E -01	0.2100000E 01
0.1456385E 02	0.3211172E 01	-0.1104886E 02	0.2200000E 01	0.1642170E 02	0.2321640E 01	-0.1054074E -01	0.2200000E 01
0.1283478E 02	0.3017027E 01	-0.1146007E 02	0.2300000E 01	0.1448190E 02	0.2014459E 01	-0.1100555E -01	0.2300000E 01
0.1130613E 02	0.2836701E 01	-0.1185940E 02	0.2400000E 01	0.1271386E 02	0.1722968E 01	-0.1141229E -01	0.2400000E 01
0.9951004E 01	0.2669709E 01	-0.1218568E 02	0.2500000E 01	0.1126112E 02	0.1484453E 01	-0.1178596E -01	0.2500000E 01
0.8750778E 01	0.2515031E 01	-0.1250161E 02	0.2600000E 01	0.9719170E 01	0.1260074E 01	-0.1212538E -01	0.2600000E 01
0.7679210E 01	0.2371704E 01	-0.1278977E 02	0.2700000E 01	0.8274965E 01	0.1025286E 01	-0.1241326E -01	0.2700000E 01
0.6720715E 01	0.2238825E 01	-0.1305249E 02	0.2800000E 01	0.7058583E 01	0.0817755E 01	-0.1271610E -01	0.2800000E 01
0.5859522E 01	0.2115556E 01	-0.1329181E -02	0.2900000E 01	0.6070215E 01	0.0624615E 01	-0.1297425E -01	0.2900000E 01
0.5083788E 01	0.2001120E 01	-0.1351005E -02	0.3000000E 01	0.5342899E 01	0.0471342E 01	-0.1329988E -01	0.3000000E 01
TENSILE END LOADS				TENSILE END LOADS			
0.1191515E 03	0.9519185E 01	0.1641255E 01	0.4000000E -00	0.1343989E 03	0.1012231E 02	0.5088922E 01	0.3000000E 00
0.1130867E 03	0.9259688E 01	0.1018450E 01	0.4000000E -00	0.1271756E 03	0.9864697E 01	0.4424560E 01	0.4000000E -00
0.1057772E 03	0.8995511E 01	0.2263622E 01	0.5000000E 00	0.1191364E 03	0.9512817E 01	0.3621776E 01	0.5000000E 00
0.9777091E 02	0.8588767E 01	0.1400344E 01	0.6000000E 00	0.1101122E 03	0.9131227E 01	0.2711788E 01	0.6000000E 00
0.8942555E 02	0.8201497E 01	0.4812687E -00	0.7000000E 00	0.1007140E 03	0.8722594E 01	0.1725429E 01	0.7000000E 00
0.8107662E 02	0.7795049E 01	-0.4888314E -00	0.8000000E 00	0.9129673E 02	0.8209089E 01	0.6938253E -00	0.8000000E 00
0.7295265E 02	0.7379568E 01	-0.1477448E 01	0.9000000E 00	0.8214158E 02	0.7849422E 01	-0.1581695E -00	0.9000000E 00
0.6524215E 02	0.6943250E 01	-0.2434201E 01	0.1000000E 01	0.7345525E 02	0.7407783E 01	-0.1407783E -01	0.1000000E 01
0.5806356E 02	0.6554120E 01	-0.1430495E 01	0.1100000E 01	0.6516528E 02	0.6972817E 01	-0.2431008E -01	0.1100000E 01
0.5148188E 02	0.6156196E 01	-0.4365210E 01	0.1200000E 01	0.5795094E 02	0.6450120E 01	-0.3432473E -01	0.1200000E 01
0.4552019E 02	0.5774268E 01	-0.5258513E 01	0.1300000E 01	0.5121625E 02	0.5944018E 01	-0.4380438E -01	0.1300000E 01
0.4017093E 02	0.5410355E 01	-0.6104257E 01	0.1400000E 01	0.4521267E 02	0.5457786E 01	-0.5285260E -01	0.1400000E 01
0.3540578E 02	0.5066154E 01	-0.6898848E 01	0.1500000E 01	0.3984825E 02	0.5027137E 01	-0.6132109E -01	0.1500000E 01
0.3110389E 02	0.4742465E 01	-0.7640710E 01	0.1600000E 01	0.3500671E 02	0.4621497E 01	-0.6923497E -01	0.1600000E 01
0.2745210E 02	0.4439181E 01	-0.8328810E 01	0.1700000E 01	0.3090425E 02	0.4226219E 01	-0.7587731E -01	0.1700000E 01
0.2417575E 02	0.4156550E 01	-0.8947246E 01	0.1800000E 01	0.2721456E 02	0.3842706E 01	-0.8192880E -01	0.1800000E 01
0.2129069E 02	0.3895327E 01	-0.9553710E 01	0.1900000E 01	0.2397238E 02	0.3444594E 01	-0.8687036E -01	0.1900000E 01
0.1875510E 02	0.3648805E 01	-0.1009422E 02	0.2000000E 01	0.2112411E 02	0.3088134E 01	-0.9444571E -01	0.2000000E 01
0.1652503E 02	0.3421975E 01	-0.1059091E 02	0.2100000E 01	0.1862700E 02	0.2640768E 01	-0.1007674E -01	0.2100000E 01
0.1456385E 02	0.3211172E 01	-0.1104886E 02	0.2200000E 01	0.1642170E 02	0.2321640E 01	-0.1054074E -01	0.2200000E 01
0.1283478E 02	0.3017027E 01	-0.1146007E 02	0.2300000E 01	0.1448190E 02	0.2014459E 01	-0.1100555E -01	0.2300000E 01
0.1130613E 02	0.2836701E 01	-0.1185940E 02	0.2400000E 01	0.1271386E 02	0.1722968E 01	-0.1141229E -01	0.2400000E 01
0.9951004E 01	0.2669709E 01	-0.1218568E 02	0.2500000E 01	0.1126112E 02	0.1484453E 01	-0.1178596E -01	0.2500000E 01
0.8750778E 01	0.2515031E 01	-0.1250161E 02	0.2600000E 01	0.9719170E 01	0.1260074E 01	-0.1212538E -01	0.2600000E 01
0.7679210E 01	0.2371704E 01	-0.1278977E 02	0.2700000E 01	0.8274965E 01	0.1025286E 01	-0.1241326E -01	0.2700000E 01
0.6720715E 01	0.22388						

TABLE 1.5-1 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_D , \bar{T} AND \bar{W}

$\bar{T}_D = -0.1600000E 02$				$\bar{W} = 0.2100000E 02$				$\bar{T}_D = -0.1600000E 02$				$\bar{W} = 0.2400000E 02$			
\bar{T}		\bar{y}	\bar{M}	$\bar{\lambda}$		\bar{T}		\bar{y}		\bar{M}	$\bar{\lambda}$	\bar{T}		\bar{y}	\bar{M}
COMPRESSIVE END LOADS															
0.1987333E 03	0.1237500E 02	0.1050000E 02	0.	0.	0.3000000E -00	0.2188190E 03	0.1300000E 02	0.1200000E 02	0.	0.	0.3000000E -00	0.2188190E 03	0.1300000E 02	0.1200000E 02	0.
0.2139823E 03	0.1285259E 02	0.1165073E 02	-0.4000000E -00	-0.4000000E -00	0.4000000E 00	0.2356155E 03	0.1350133E 02	0.1321512E 02	-0.3000000E -00	-0.3000000E -00	0.3000000E 00	0.2356155E 03	0.1350133E 02	0.1321512E 02	-0.3000000E -00
0.2270987E 03	0.1324986E 02	0.1261998E 02	-0.4000000E -00	-0.4000000E -00	0.4000000E 00	0.2500641E 03	0.1391835E 02	0.1422694E 02	-0.4000000E -00	-0.4000000E -00	0.4000000E 00	0.2500641E 03	0.1391835E 02	0.1422694E 02	-0.4000000E -00
0.2458161E 03	0.1379757E 02	0.1394939E 02	-0.5000000E -00	-0.5000000E -00	0.5000000E 00	0.2706827E 03	0.1449328E 02	0.1562332E 02	-0.5000000E -00	-0.5000000E -00	0.5000000E 00	0.2706827E 03	0.1449328E 02	0.1562332E 02	-0.5000000E -00
0.2720242E 03	0.1453066E 02	0.1573104E 02	-0.6000000E -00	-0.6000000E -00	0.6000000E 00	0.2995519E 03	0.1526280E 02	0.1749461E 02	-0.6000000E -00	-0.6000000E -00	0.6000000E 00	0.2995519E 03	0.1526280E 02	0.1749461E 02	-0.6000000E -00
0.3088045E 03	0.1550239E 02	0.1809617E 02	-0.7000000E -00	-0.7000000E -00	0.7000000E 00	0.3300679E 03	0.1628201E 02	0.1997858E 02	-0.7000000E -00	-0.7000000E -00	0.7000000E 00	0.3300679E 03	0.1628201E 02	0.1997858E 02	-0.7000000E -00
0.3613664E 03	0.1679572E 02	0.2124926E 02	-0.8000000E -00	-0.8000000E -00	0.8000000E 00	0.3796977E 03	0.1764038E 02	0.2328894E 02	-0.8000000E -00	-0.8000000E -00	0.8000000E 00	0.3796977E 03	0.1764038E 02	0.2328894E 02	-0.8000000E -00
0.4390142E 03	0.1854493E 02	0.2552139E 02	-0.9000000E -00	-0.9000000E -00	0.9000000E 00	0.4835074E 03	0.1947646E 02	0.2777593E 02	-0.9000000E -00	-0.9000000E -00	0.9000000E 00	0.4835074E 03	0.1947646E 02	0.2777593E 02	-0.9000000E -00
0.5596771E 03	0.2098018E 02	0.3148018E 02	-0.1000000E 01	-0.1000000E 01	0.1000000E 01	0.6164327E 03	0.2203263E 02	0.3403263E 02	-0.1000000E 01	-0.1000000E 01	0.1000000E 01	0.6164327E 03	0.2203263E 02	0.3403263E 02	-0.1000000E 01
0.7616966E 03	0.2452895E 02	0.4018000E 02	-0.1000000E 01	-0.1000000E 01	0.1000000E 01	0.8389855E 03	0.2575756E 02	0.4316635E 02	-0.1000000E 01	-0.1000000E 01	0.1000000E 01	0.8389855E 03	0.2575756E 02	0.4316635E 02	-0.1000000E 01
0.1140121E 04	0.3000167E 02	0.5381766E 02	-0.1000000E 01	-0.1000000E 01	0.1000000E 01	0.1255077E 04	0.3155897E 02	0.5748865E 02	-0.1000000E 01	-0.1000000E 01	0.1000000E 01	0.1255077E 04	0.3155897E 02	0.5748865E 02	-0.1000000E 01
0.1990043E 04	0.3984614E 02	0.7763998E 02	-0.1000000E 01	-0.1000000E 01	0.1000000E 01	0.2192275E 04	0.4183847E 02	0.8270094E 02	-0.1000000E 01	-0.1000000E 01	0.1000000E 01	0.2192275E 04	0.4183847E 02	0.8270094E 02	-0.1000000E 01
0.4669056E 04	0.6120354E 02	0.1504509E 03	-0.1000000E 01	-0.1000000E 01	0.1000000E 01	0.5145583E 04	0.6425187E 02	0.1579337E 03	-0.1000000E 01	-0.1000000E 01	0.1000000E 01	0.5145583E 04	0.6425187E 02	0.1579337E 03	-0.1000000E 01
0.9026074E 04	0.9522039E 02	0.1896759E 03	-0.1000000E 01	-0.1000000E 01	0.1000000E 01	0.9945157E 04	0.6946016E 02	0.2000900E 03	-0.1000000E 01	-0.1000000E 01	0.1000000E 01	0.9945157E 04	0.6946016E 02	0.2000900E 03	-0.1000000E 01
0.2542509E 05	0.1412944E 03	0.3327798E 03	-0.1500000E 01	-0.1500000E 01	0.1500000E 01	0.2800987E 05	0.1503625E 03	0.3503155E 03	-0.1500000E 01	-0.1500000E 01	0.1500000E 01	0.2800987E 05	0.1503625E 03	0.3503155E 03	-0.1500000E 01
0.2852662E 06	0.4805498E 03	0.1165021E 04	-0.1500000E 01	-0.1500000E 01	0.1500000E 01	0.3142689E 06	0.5043999E 03	0.1223821E 04	-0.1500000E 01	-0.1500000E 01	0.1500000E 01	0.3142689E 06	0.5043999E 03	0.1223821E 04	-0.1500000E 01
TENSILE END LOADS															
0.1850813E 03	0.1193111E 02	0.9626704E 01	0.3000000E -00	0.3000000E -00	0.3000000E -00	0.2037852E 03	0.1255404E 02	0.1087194E 02	0.3000000E -00	0.3000000E -00	0.3000000E -00	0.2037852E 03	0.1255404E 02	0.1087194E 02	0.3000000E -00
0.1753823E 03	0.1160893E 02	0.8642891E 01	0.4000000E -00	0.4000000E -00	0.4000000E -00	0.1930978E 03	0.1219374E 02	0.1004900E 02	0.4000000E -00	0.4000000E -00	0.4000000E -00	0.1930978E 03	0.1219374E 02	0.1004900E 02	0.4000000E -00
0.1610194E 03	0.1121474E 02	0.7896311E 01	0.5000000E -00	0.5000000E -00	0.5000000E -00	0.1805822E 03	0.1178204E 02	0.9054489E 01	0.5000000E -00	0.5000000E -00	0.5000000E -00	0.1805822E 03	0.1178204E 02	0.9054489E 01	0.5000000E -00
0.1515750E 03	0.1076939E 02	0.6673021E 01	0.6000000E -00	0.6000000E -00	0.6000000E -00	0.1668756E 03	0.1131454E 02	0.7926765E 01	0.6000000E -00	0.6000000E -00	0.6000000E -00	0.1668756E 03	0.1131454E 02	0.7926765E 01	0.6000000E -00
0.1366159E 03	0.1028521E 02	0.5459910E 01	0.7000000E -00	0.7000000E -00	0.7000000E -00	0.1526023E 03	0.1080700E 02	0.6704571E 01	0.7000000E -00	0.7000000E -00	0.7000000E -00	0.1526023E 03	0.1080700E 02	0.6704571E 01	0.7000000E -00
0.1256323E 03	0.9778455E 01	0.4251795E 01	0.8000000E -00	0.8000000E -00	0.8000000E -00	0.1380256E 03	0.1027429E 02	0.5424452E 01	0.8000000E -00	0.8000000E -00	0.8000000E -00	0.1380256E 03	0.1027429E 02	0.5424452E 01	0.8000000E -00
0.1130123E 03	0.9257669E 01	0.2997668E 01	0.9000000E -00	0.9000000E -00	0.9000000E -00	0.1244018E 03	0.9729695E 01	0.4118974E 01	0.9000000E -00	0.9000000E -00	0.9000000E -00	0.1244018E 03	0.9729695E 01	0.4118974E 01	0.9000000E -00
0.1010364E 03	0.8747711E 01	0.1759729E 01	0.1000000E 01	0.1000000E 01	0.1000000E 01	0.1112173E 03	0.9184444E 01	0.2815568E 01	0.1000000E 01	0.1000000E 01	0.1000000E 01	0.1112173E 03	0.9184444E 01	0.2815568E 01	0.1000000E 01
0.8989480E 02	0.8226803E 01	0.5630515E 00	0.1100000E 01	0.1100000E 01	0.1100000E 01	0.9894610E 02	0.8647572E 01	0.1516438E 01	0.1100000E 01	0.1100000E 01	0.1100000E 01	0.9894610E 02	0.8647572E 01	0.1516438E 01	0.1100000E 01
0.7968271E 02	0.7732128E 01	-0.6362611E 00	0.1200000E 01	0.1200000E 01	0.1200000E 01	0.8770150E 02	0.8126050E 01	0.7964761E 00	0.1200000E 01	0.1200000E 01	0.1200000E 01	0.8770150E 02	0.8126050E 01	0.7964761E 00	0.1200000E 01
0.7053795E 02	0.7254799E 01	-0.1760611E 01	0.1300000E 01	0.1300000E 01	0.1300000E 01	0.7752311E 02	0.7624912E 01	0.7187588E 01	0.1300000E 01	0.1300000E 01	0.1300000E 01	0.7752311E 02	0.7624912E 01	0.7187588E 01	0.1300000E 01
0.6214866E 02	0.6800137E 01	-0.2020269E 01	0.1400000E 01	0.1400000E 01	0.1400000E 01	0.6869757E 02	0.6697657E 01	0.6696049E 01	0.1400000E 01	0.1400000E 01	0.1400000E 01	0.6869757E 02	0.6697657E 01	0.6696049E 01	0.1400000E 01
0.5477071E 02	0.6357088E 01	0.4832693E 01	0.1500000E 01	0.1500000E 01	0.1500000E 01	0.6027657E 02	0.6027657E 01	0.6027657E 01	0.1500000E 01	0.1500000E 01	0.1500000E 01	0.6027657E 02	0.6027657E 01	0.6027657E 01	0.1500000E 01
0.4802022E 02	0.5965569E 01	0.4771857E 01	0.1600000E 01	0.1600000E 01	0.1600000E 01	0.5300955E 02	0.5300955E 01	0.5300955E 01	0.1600000E 01	0.1600000E 01	0.1600000E 01	0.5300955E 02	0.5300955E 01	0.5300955E 01	0.1600000E 01
0.4240286E 02	0.5580734E 01	0.5645641E 01	0.1700000E 01	0.1700000E 01	0.1700000E 01	0.4675467E 02	0.4675467E 01	0.4675467E 01	0.1700000E 01	0.1700000E 01	0.1700000E 01	0.4675467E 02	0.4675467E 01	0.4675467E 01	0.1700000E 01
0.3742086E 02	0.5233152E 01	0.6455413E 01	0.1800000E 01	0.1800000E 01	0.1800000E 01	0.4118625E 02	0.4118625E 01	0.4118625E 01	0.1800000E 01	0.1800000E 01	0.1800000E 01	0.4118625E 02	0.4118625E 01	0.4118625E 01	0.1800000E 01
0.3297351E 02	0.4903993E 01	0.7705151E 01	0.1900000E 01	0.1900000E 01	0.1900000E 01	0.3629973E 02	0.3629973E 01	0.3629973E 01	0.1900000E 01	0.1900000E 01	0.1900000E 01	0.3629973E 02	0.3629973E 01	0.3629973E 01	0.1900000E 01
0.2908002E 02	0.4598157E 01	0.7892626E 01	0.2000000E 01	0.2000000E 01	0.2000000E 01	0.3201496E 02	0.3201496E 01	0.3201496E 01	0.2000000E 01	0.2000000E 01	0.2000000E 01	0.3201496E 02	0.3201496E 01	0.3201496E 01	0.2000000E 01
0.2566123E 02	0.4311477E 01	0.8526404E 01	0.2100000E 01	0.2100000E 01	0.2100000E 01	0.2825789E 02	0.2825789E 01	0.2825789E 01	0.2100000E 01	0.2100000E 01	0.2100000E 01	0.2825789E 02	0.2825789E 01	0.2825789E 01	0.2100000E 01
0.2266011E 02	0.4051106E 01	0.9108133E 01	0.2200000E 01	0.2200000E 01	0.2200000E 01	0.2496152E 02	0.2496152E 01	0.2496152E 01	0.2200000E 01	0.2200000E 01	0.2200000E 01	0.2496152E 02			

TABLE 1.5-1 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_d , \bar{T} AND \bar{W}

\bar{T}_D	\bar{W}	\bar{W}	\bar{T}_D	\bar{W}	\bar{W}
\bar{y}	\bar{M}	\bar{X}	\bar{y}	\bar{M}	\bar{X}
COMPRESSIVE END LOADS			COMPRESSIVE END LOADS		
0.1672762E 03	0.1125000E 02	0.3000000E 01	0.1857008E 03	0.1187500E 02	0.4500000E 01
0.1800771E 03	0.1168773E 02	0.4051800E 01	0.1999001E 03	0.1233547E 02	0.5610192E 01
0.1910479E 03	0.1205003E 02	0.4928005E 01	0.2121170E 03	0.1271852E 02	0.6514963E 01
0.2067329E 03	0.1255091E 02	0.6137733E 01	0.2295401E 03	0.1324666E 02	0.7811660E 01
0.2286938E 03	0.1322140E 02	0.7759704E 01	0.2539406E 03	0.1393554E 02	0.9523274E 01
0.2595115E 03	0.1411019E 02	0.9913993E 01	0.2881928E 03	0.1469000E 02	0.1177660E 02
0.3035805E 03	0.1529320E 02	0.1278765E 02	0.3317277E 03	0.1579861E 02	0.1462137E 02
0.3508791E 03	0.1689315E 02	0.1640134E 02	0.4094131E 03	0.1782886E 02	0.1893511E 02
0.4696770E 03	0.2191221E 02	0.2212112E 02	0.5217405E 03	0.2017366E 02	0.2467166E 02
0.6380972E 03	0.2726804E 02	0.3006553E 02	0.7097946E 03	0.2395666E 02	0.3305195E 02
0.9586653E 03	0.2784373E 02	0.4252617E 02	0.1062046E 04	0.2805993E 02	0.4619272E 02
0.1667749E 04	0.3658377E 02	0.6480550E 02	0.1853181E 04	0.3817720E 02	0.6934935E 02
0.3911532E 04	0.5592819E 02	0.1126119E 03	0.4346786E 04	0.5897652E 02	0.1200949E 03
0.7580695E 04	0.7700202E 02	0.1647990E 03	0.8402217E 04	0.8214600E 02	0.1721164E 03
0.1295947E 05	0.1310081E 03	0.2797682E 03	0.2366631E 05	0.1181262E 03	0.3152640E 03
0.21896217E 06	0.4398689E 03	0.1059337E 04	0.2655237E 06	0.4635492E 03	0.1118177E 04

TENSILE END LOADS				TENSILE END LOADS			
0.15581086	03	0.10884101	02	0.20280311	01	0.50000000	-00
0.14770656	03	0.10574769	02	0.13123702	01	0.46395976	03
0.13617963	03	0.10189102	02	0.45272758	00	0.11884703	02
0.12773466	03	0.97819188	01	-0.52119746	00	0.27184505	02
0.11688111	03	0.93399381	01	-0.15765708	01	0.10375612	02
0.10599366	03	0.80760756	01	-0.26806380	01	0.10327092	02
0.95409171	02	0.86019122	01	-0.10055494	01	0.08610400	01
0.85363652	02	0.79272302	01	-0.49272302	01	-0.33190911	-00
0.76011322	02	0.78599558	01	-0.60265451	01	-0.14960321	01
0.67841392	02	0.70061131	01	-0.70088021	01	-0.26862701	01
0.59678262	02	0.65701036	01	-0.10034736	01	-0.15871401	01
0.52713272	02	0.61584811	01	-0.90313465	01	-0.50331159	01
0.46512176	02	0.57622721	01	-0.99650071	01	-0.67232702	01
0.41019266	02	0.53929731	01	-0.10060012	02	-0.78899791	01
0.36172751	02	0.50047259	01	-0.15866561	02	-0.82285456	01
0.31907761	02	0.47264871	01	-0.12307792	02	-0.91984361	01
0.28160166	02	0.44284492	01	-0.12972426	02	-0.10088081	02
0.24869016	02	0.41365651	02	-0.13526462	02	-0.12915581	02
0.21978181	02	0.38810821	01	-0.14181681	02	-0.11540031	02
0.19436672	02	0.36471851	01	-0.16653351	02	-0.12108351	02
0.17190152	02	0.34254271	01	-0.15120512	02	-0.11011831	02
0.15225492	02	0.32199111	01	-0.15586692	02	-0.16468271	02
0.13009092	02	0.30298702	01	-0.15911721	02	-0.18659961	02
0.11913572	02	0.28513572	01	-0.12880992	02	-0.19155812	02
0.10555788	02	0.26890171	01	-0.16610952	02	-0.15513191	02
0.91508501	01	0.25350776	01	-0.16903932	02	-0.19104681	02
0.82127584	01	0.23980402	01	-0.17170551	02	-0.16259491	02
0.72850421	01	0.22768110	01	-0.17812991	02	-0.16567701	02
						-0.18852491	02
						-0.17127712	02

I _D = -0.20000000 02				I _D = 0.20000000 02				I _D = -0.20000000 02				I _D = 0.20000000 02			
X	Y	M	N	X	Y	M	N	X	Y	M	N	X	Y	M	N
COMPRESSIVE EMO LOADS								COMPRESSIVE EMO LOADS							
0.2051048E 01	0.1250000E 02	0.5000000E 01	0.	0.2258167E 01	0.1157200E 02	0.5000000E 01	0.	0.2258167E 01	0.1157200E 02	0.5000000E 01	0.	0.2258167E 01	0.1157200E 02	0.5000000E 01	0.
0.2288000E 01	0.1298281E 02	0.5000000E 01	0.	0.2468579E 01	0.1247200E 02	0.5000000E 01	0.	0.2468579E 01	0.1247200E 02	0.5000000E 01	0.	0.2468579E 01	0.1247200E 02	0.5000000E 01	0.
0.2582987E 03	0.1358101E 02	0.5000000E 01	0.	0.2678007E 01	0.1297200E 02	0.5000000E 01	0.	0.2678007E 01	0.1297200E 02	0.5000000E 01	0.	0.2678007E 01	0.1297200E 02	0.5000000E 01	0.
0.2955827E 03	0.1528255E 02	0.5000000E 01	0.	0.276780E 01	0.1347200E 02	0.5000000E 01	0.	0.276780E 01	0.1347200E 02	0.5000000E 01	0.	0.276780E 01	0.1347200E 02	0.5000000E 01	0.
0.2805547E 01	0.1128685E 02	0.5000000E 01	0.	0.286780E 01	0.1397200E 02	0.5000000E 01	0.	0.286780E 01	0.1397200E 02	0.5000000E 01	0.	0.286780E 01	0.1397200E 02	0.5000000E 01	0.
0.5181828E 01	0.1587102E 02	0.5000000E 01	0.	0.296780E 01	0.1447200E 02	0.5000000E 01	0.	0.296780E 01	0.1447200E 02	0.5000000E 01	0.	0.296780E 01	0.1447200E 02	0.5000000E 01	0.
0.5778179E 01	0.1682825E 02	0.5000000E 01	0.	0.306780E 01	0.1497200E 02	0.5000000E 01	0.	0.306780E 01	0.1497200E 02	0.5000000E 01	0.	0.306780E 01	0.1497200E 02	0.5000000E 01	0.
0.6572165E 01	0.1675878E 02	0.5000000E 01	0.	0.316780E 01	0.1547200E 02	0.5000000E 01	0.	0.316780E 01	0.1547200E 02	0.5000000E 01	0.	0.316780E 01	0.1547200E 02	0.5000000E 01	0.
0.5745500E 01	0.1675878E 02	0.5000000E 01	0.	0.326780E 01	0.1597200E 02	0.5000000E 01	0.	0.326780E 01	0.1597200E 02	0.5000000E 01	0.	0.326780E 01	0.1597200E 02	0.5000000E 01	0.
0.7064521E 03	0.2082527E 02	0.5000000E 01	0.	0.336780E 01	0.1647200E 02	0.5000000E 01	0.	0.336780E 01	0.1647200E 02	0.5000000E 01	0.	0.336780E 01	0.1647200E 02	0.5000000E 01	0.
0.7118299E 04	0.2082527E 02	0.5000000E 01	0.	0.346780E 01	0.1697200E 02	0.5000000E 01	0.	0.346780E 01	0.1697200E 02	0.5000000E 01	0.	0.346780E 01	0.1697200E 02	0.5000000E 01	0.
0.2086805E 04	0.2082527E 02	0.5000000E 01	0.	0.356780E 01	0.1747200E 02	0.5000000E 01	0.	0.356780E 01	0.1747200E 02	0.5000000E 01	0.	0.356780E 01	0.1747200E 02	0.5000000E 01	0.
0.4805007E 04	0.2082527E 02	0.5000000E 01	0.	0.366780E 01	0.1797200E 02	0.5000000E 01	0.	0.366780E 01	0.1797200E 02	0.5000000E 01	0.	0.366780E 01	0.1797200E 02	0.5000000E 01	0.
0.7288190E 04	0.2082527E 02	0.5000000E 01	0.	0.376780E 01	0.1847200E 02	0.5000000E 01	0.	0.376780E 01	0.1847200E 02	0.5000000E 01	0.	0.376780E 01	0.1847200E 02	0.5000000E 01	0.
0.7818259E 04	0.2082527E 02	0.5000000E 01	0.	0.386780E 01	0.1897200E 02	0.5000000E 01	0.	0.386780E 01	0.1897200E 02	0.5000000E 01	0.	0.386780E 01	0.1897200E 02	0.5000000E 01	0.
0.2955298E 08	0.4871897E 01	0.1178977E 04	0.	0.396780E 01	0.1947200E 02	0.5000000E 01	0.	0.396780E 01	0.1947200E 02	0.5000000E 01	0.	0.396780E 01	0.1947200E 02	0.5000000E 01	0.

TENSILE END LOADS				TENSILE END LOADS			
0.19101684 01	0.17067916 02	0.49155031 01	0.10000000 02	0.21000000 03	0.12852391 02	0.24161210 01	0.19030001 02
0.18101721 01	0.17171111 02	0.41255031 01	0.10000000 02	0.19901394 01	0.12101321 02	0.25101701 01	0.19000000 01
0.16937692 01	0.11521111 02	0.11601911 01	0.10000000 02	0.12611701 01	0.11191021 02	0.24522761 01	0.19030001 02
0.15057609 01	0.10872751 02	0.00095911 01	0.00000000 00	0.11512001 01	0.11151701 02	0.11091731 01	0.19000000 01
0.14122526 01	0.10102111 02	0.91275111 02	0.10000000 02	0.15271161 01	0.10901281 02	0.21572111 01	0.19000000 01
0.12981594 01	0.98617711 02	0.15137501 00	0.00000000 00	0.13571111 01	0.10161621 02	0.86127611 00	0.19000000 01
0.11686721 01	0.95119411 02	0.15669911 01	0.10000000 02	0.12821211 01	0.98119901 02	0.86771611 00	0.19000000 01
0.10515191 01	0.88155601 02	0.28115601 01	0.10000000 02	0.11000000 01	0.92557201 01	0.11972701 01	0.10000000 01
0.91066460 02	0.87297131 01	0.00197721 01	0.11000000 01	0.10200000 01	0.10200000 01	0.87180211 01	0.11000000 01
0.20566591 02	0.77971701 02	0.52221101 01	0.12000000 01	0.10000000 01	0.10000000 01	0.87180211 01	0.11000000 01
0.11027271 02	0.71501601 02	0.61555271 01	0.11000000 01	0.10000000 01	0.10000000 01	0.87180211 01	0.11000000 01
0.68686752 02	0.68675271 02	0.74257271 01	0.10000000 01	0.10000000 01	0.10000000 01	0.87180211 01	0.11000000 01
0.56887511 02	0.66151091 02	0.84119711 01	0.11000000 01	0.10000000 01	0.10000000 01	0.87180211 01	0.11000000 01
0.1540081 02	0.62051091 02	0.93715501 01	0.10000000 01	0.10000000 01	0.10000000 01	0.87180211 01	0.11000000 01
0.06221391 01	0.56208751 01	0.10265501 02	0.11000000 01	0.10000000 01	0.10000000 01	0.87180211 01	0.11000000 01
0.19005201 02	0.52420901 01	0.11052071 02	0.18000000 01	0.10000000 01	0.10000000 01	0.87180211 01	0.11000000 01
0.19621051 02	0.49290751 01	0.11796571 02	0.19000000 01	0.10000000 01	0.10000000 01	0.87180211 01	0.11000000 01
0.10402471 02	0.46201001 01	0.12481161 02	0.20000000 01	0.10000000 01	0.10000000 01	0.87180211 01	0.11000000 01
0.26076601 02	0.45132601 01	0.17109801 02	0.21000000 01	0.10000000 01	0.10000000 01	0.87180211 01	0.11000000 01
0.23776491 02	0.40671621 01	0.16850501 02	0.22000000 01	0.10000000 01	0.10000000 01	0.87180211 01	0.11000000 01
0.21052711 02	0.38765971 01	0.16211161 02	0.23000000 01	0.10000000 01	0.10000000 01	0.87180211 01	0.11000000 01
0.18653811 02	0.35921511 01	0.16679711 02	0.24000000 01	0.10000000 01	0.10000000 01	0.87180211 01	0.11000000 01
0.16515601 02	0.33810671 01	0.16157671 02	0.25000000 01	0.10000000 01	0.10000000 01	0.87180211 01	0.11000000 01
0.14661211 02	0.31872701 01	0.15557571 02	0.26000000 01	0.10000000 01	0.10000000 01	0.87180211 01	0.11000000 01
0.13012471 02	0.30018601 01	0.14980001 02	0.27000000 01	0.10000000 01	0.10000000 01	0.87180211 01	0.11000000 01
0.12221711 02	0.28221711 01	0.14231511 02	0.28000000 01	0.10000000 01	0.10000000 01	0.87180211 01	0.11000000 01
0.10201751 02	0.26795971 01	0.16515571 02	0.29000000 01	0.10000000 01	0.10000000 01	0.87180211 01	0.11000000 01

TABLE 1.5-1 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_d , \bar{T} AND \bar{W}

[illegible]

TABLE 1.5-2
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{m} FOR SPECIFIED VALUES OF \bar{T}_d , \bar{T} AND \bar{Q}
(Pages 1.38 through 1.55)

TABLE 1.5-2 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_d , \bar{T} AND \bar{Q}

\bar{T}_d	\bar{y}	\bar{M}	$\bar{\lambda}$	\bar{T}_d	\bar{y}	\bar{M}	$\bar{\lambda}$
COMPRESSIVE END LOADS				COMPRESSIVE END LOADS			
0.	0.	0.	0.	0.12000000	0.	0.	0.
0.15000000	0.	0.	0.	0.13000000	0.	0.	0.
0.20000000	0.	0.	0.	0.14000000	0.	0.	0.
0.25000000	0.	0.	0.	0.15000000	0.	0.	0.
0.30000000	0.	0.	0.	0.16000000	0.	0.	0.
0.35000000	0.	0.	0.	0.17000000	0.	0.	0.
0.40000000	0.	0.	0.	0.18000000	0.	0.	0.
0.45000000	0.	0.	0.	0.19000000	0.	0.	0.
0.50000000	0.	0.	0.	0.20000000	0.	0.	0.
0.55000000	0.	0.	0.	0.21000000	0.	0.	0.
0.60000000	0.	0.	0.	0.22000000	0.	0.	0.
0.65000000	0.	0.	0.	0.23000000	0.	0.	0.
0.70000000	0.	0.	0.	0.24000000	0.	0.	0.
0.75000000	0.	0.	0.	0.25000000	0.	0.	0.
0.80000000	0.	0.	0.	0.26000000	0.	0.	0.
0.85000000	0.	0.	0.	0.27000000	0.	0.	0.
0.90000000	0.	0.	0.	0.28000000	0.	0.	0.
0.95000000	0.	0.	0.	0.29000000	0.	0.	0.
1.00000000	0.	0.	0.	0.30000000	0.	0.	0.
1.05000000	0.	0.	0.	0.31000000	0.	0.	0.
1.10000000	0.	0.	0.	0.32000000	0.	0.	0.
1.15000000	0.	0.	0.	0.33000000	0.	0.	0.
1.20000000	0.	0.	0.	0.34000000	0.	0.	0.
1.25000000	0.	0.	0.	0.35000000	0.	0.	0.
1.30000000	0.	0.	0.	0.36000000	0.	0.	0.
1.35000000	0.	0.	0.	0.37000000	0.	0.	0.
1.40000000	0.	0.	0.	0.38000000	0.	0.	0.
1.45000000	0.	0.	0.	0.39000000	0.	0.	0.
1.50000000	0.	0.	0.	0.40000000	0.	0.	0.
1.55000000	0.	0.	0.	0.41000000	0.	0.	0.
1.60000000	0.	0.	0.	0.42000000	0.	0.	0.
1.65000000	0.	0.	0.	0.43000000	0.	0.	0.
1.70000000	0.	0.	0.	0.44000000	0.	0.	0.
1.75000000	0.	0.	0.	0.45000000	0.	0.	0.
1.80000000	0.	0.	0.	0.46000000	0.	0.	0.
1.85000000	0.	0.	0.	0.47000000	0.	0.	0.
1.90000000	0.	0.	0.	0.48000000	0.	0.	0.
1.95000000	0.	0.	0.	0.49000000	0.	0.	0.
2.00000000	0.	0.	0.	0.50000000	0.	0.	0.
2.05000000	0.	0.	0.	0.51000000	0.	0.	0.
2.10000000	0.	0.	0.	0.52000000	0.	0.	0.
2.15000000	0.	0.	0.	0.53000000	0.	0.	0.
2.20000000	0.	0.	0.	0.54000000	0.	0.	0.
2.25000000	0.	0.	0.	0.55000000	0.	0.	0.
2.30000000	0.	0.	0.	0.56000000	0.	0.	0.
2.35000000	0.	0.	0.	0.57000000	0.	0.	0.
2.40000000	0.	0.	0.	0.58000000	0.	0.	0.
2.45000000	0.	0.	0.	0.59000000	0.	0.	0.
2.50000000	0.	0.	0.	0.60000000	0.	0.	0.
2.55000000	0.	0.	0.	0.61000000	0.	0.	0.
2.60000000	0.	0.	0.	0.62000000	0.	0.	0.
2.65000000	0.	0.	0.	0.63000000	0.	0.	0.
2.70000000	0.	0.	0.	0.64000000	0.	0.	0.
2.75000000	0.	0.	0.	0.65000000	0.	0.	0.
2.80000000	0.	0.	0.	0.66000000	0.	0.	0.
2.85000000	0.	0.	0.	0.67000000	0.	0.	0.
2.90000000	0.	0.	0.	0.68000000	0.	0.	0.
2.95000000	0.	0.	0.	0.69000000	0.	0.	0.
3.00000000	0.	0.	0.	0.70000000	0.	0.	0.
3.05000000	0.	0.	0.	0.71000000	0.	0.	0.
3.10000000	0.	0.	0.	0.72000000	0.	0.	0.
3.15000000	0.	0.	0.	0.73000000	0.	0.	0.
3.20000000	0.	0.	0.	0.74000000	0.	0.	0.
3.25000000	0.	0.	0.	0.75000000	0.	0.	0.
3.30000000	0.	0.	0.	0.76000000	0.	0.	0.
3.35000000	0.	0.	0.	0.77000000	0.	0.	0.
3.40000000	0.	0.	0.	0.78000000	0.	0.	0.
3.45000000	0.	0.	0.	0.79000000	0.	0.	0.
3.50000000	0.	0.	0.	0.80000000	0.	0.	0.
3.55000000	0.	0.	0.	0.81000000	0.	0.	0.
3.60000000	0.	0.	0.	0.82000000	0.	0.	0.
3.65000000	0.	0.	0.	0.83000000	0.	0.	0.
3.70000000	0.	0.	0.	0.84000000	0.	0.	0.
3.75000000	0.	0.	0.	0.85000000	0.	0.	0.
3.80000000	0.	0.	0.	0.86000000	0.	0.	0.
3.85000000	0.	0.	0.	0.87000000	0.	0.	0.
3.90000000	0.	0.	0.	0.88000000	0.	0.	0.
3.95000000	0.	0.	0.	0.89000000	0.	0.	0.
4.00000000	0.	0.	0.	0.90000000	0.	0.	0.
4.05000000	0.	0.	0.	0.91000000	0.	0.	0.
4.10000000	0.	0.	0.	0.92000000	0.	0.	0.
4.15000000	0.	0.	0.	0.93000000	0.	0.	0.
4.20000000	0.	0.	0.	0.94000000	0.	0.	0.
4.25000000	0.	0.	0.	0.95000000	0.	0.	0.
4.30000000	0.	0.	0.	0.96000000	0.	0.	0.
4.35000000	0.	0.	0.	0.97000000	0.	0.	0.
4.40000000	0.	0.	0.	0.98000000	0.	0.	0.
4.45000000	0.	0.	0.	0.99000000	0.	0.	0.
4.50000000	0.	0.	0.	1.00000000	0.	0.	0.

TABLE 1.5-2 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_d , \bar{T} AND \bar{Q}

$\bar{T}_D = 0$				$\bar{T}_D = 0$				$\bar{T}_D = 0$							
\bar{T}		\bar{y}		\bar{M}		\bar{A}		\bar{T}		\bar{y}		\bar{M}		\bar{A}	
COMPRESSIVE END LOADS								COMPRESSIVE END LOADS							
0.1920000E 02	0.4000000E 01	0.1220000E 02	0.	0.	0.1000000E -00	0.	0.	0.1920000E 02	0.4000000E 01	0.1220000E 02	0.	0.	0.1000000E -00	0.	0.
0.2069885E 02	0.4189788E 01	0.1217385E 02	-0.1000000E -00	-0.1000000E -00	-0.1000000E -00	-0.1000000E -00	-0.1000000E -00	0.2069885E 02	0.4189788E 01	0.1217385E 02	-0.1000000E -00	-0.1000000E -00	-0.1000000E -00	-0.1000000E -00	-0.1000000E -00
0.2197306E 02	0.4273770E 01	0.1216030E 02	-0.4000000E -00	-0.4000000E -00	-0.4000000E -00	-0.4000000E -00	-0.4000000E -00	0.2197306E 02	0.4273770E 01	0.1216030E 02	-0.4000000E -00	-0.4000000E -00	-0.4000000E -00	-0.4000000E -00	-0.4000000E -00
0.2380916E 02	0.4455038E 01	0.1211126E 02	-0.5000000E -00	-0.5000000E -00	-0.5000000E -00	-0.5000000E -00	-0.5000000E -00	0.2380916E 02	0.4455038E 01	0.1211126E 02	-0.5000000E -00	-0.5000000E -00	-0.5000000E -00	-0.5000000E -00	-0.5000000E -00
0.2637085E 02	0.4618266E 01	0.1168273E 02	-0.6000000E -00	-0.6000000E -00	-0.6000000E -00	-0.6000000E -00	-0.6000000E -00	0.2637085E 02	0.4618266E 01	0.1168273E 02	-0.6000000E -00	-0.6000000E -00	-0.6000000E -00	-0.6000000E -00	-0.6000000E -00
0.2998192E 02	0.4918019E 01	0.1188172E 02	-0.7000000E -00	-0.7000000E -00	-0.7000000E -00	-0.7000000E -00	-0.7000000E -00	0.2998192E 02	0.4918019E 01	0.1188172E 02	-0.7000000E -00	-0.7000000E -00	-0.7000000E -00	-0.7000000E -00	-0.7000000E -00
0.3502907E 02	0.5182151E 01	0.1158858E 02	-0.8000000E -00	-0.8000000E -00	-0.8000000E -00	-0.8000000E -00	-0.8000000E -00	0.3502907E 02	0.5182151E 01	0.1158858E 02	-0.8000000E -00	-0.8000000E -00	-0.8000000E -00	-0.8000000E -00	-0.8000000E -00
0.4265586E 02	0.5928503E 01	0.1180211E 02	-0.9000000E -00	-0.9000000E -00	-0.9000000E -00	-0.9000000E -00	-0.9000000E -00	0.4265586E 02	0.5928503E 01	0.1180211E 02	-0.9000000E -00	-0.9000000E -00	-0.9000000E -00	-0.9000000E -00	-0.9000000E -00
0.5440196E 02	0.6688893E 01	0.1160889E 02	-0.1000000E -01	-0.1000000E -01	-0.1000000E -01	-0.1000000E -01	-0.1000000E -01	0.5440196E 02	0.6688893E 01	0.1160889E 02	-0.1000000E -01	-0.1000000E -01	-0.1000000E -01	-0.1000000E -01	-0.1000000E -01
0.7005786E 02	0.7766801E 01	0.1218337E 02	-0.1100000E -01	-0.1100000E -01	-0.1100000E -01	-0.1100000E -01	-0.1100000E -01	0.7005786E 02	0.7766801E 01	0.1218337E 02	-0.1100000E -01	-0.1100000E -01	-0.1100000E -01	-0.1100000E -01	-0.1100000E -01
0.1108506E 03	0.9528840E 01	0.1257215E 02	-0.1200000E -01	-0.1200000E -01	-0.1200000E -01	-0.1200000E -01	-0.1200000E -01	0.1108506E 03	0.9528840E 01	0.1257215E 02	-0.1200000E -01	-0.1200000E -01	-0.1200000E -01	-0.1200000E -01	-0.1200000E -01
0.1938496E 03	0.1257407E 02	0.1332518E 02	-0.1300000E -01	-0.1300000E -01	-0.1300000E -01	-0.1300000E -01	-0.1300000E -01	0.1938496E 03	0.1257407E 02	0.1332518E 02	-0.1300000E -01	-0.1300000E -01	-0.1300000E -01	-0.1300000E -01	-0.1300000E -01
0.4537268E 03	0.1923273E 02	0.1489814E 02	-0.1400000E -01	-0.1400000E -01	-0.1400000E -01	-0.1400000E -01	-0.1400000E -01	0.4537268E 03	0.1923273E 02	0.1489814E 02	-0.1400000E -01	-0.1400000E -01	-0.1400000E -01	-0.1400000E -01	-0.1400000E -01
0.8768571E 03	0.2871929E 02	0.2681773E 02	-0.1500000E -01	-0.1500000E -01	-0.1500000E -01	-0.1500000E -01	-0.1500000E -01	0.8768571E 03	0.2871929E 02	0.2681773E 02	-0.1500000E -01	-0.1500000E -01	-0.1500000E -01	-0.1500000E -01	-0.1500000E -01
0.2470024E 04	0.4480103E 02	0.1125114E 03	-0.1500000E -01	-0.1500000E -01	-0.1500000E -01	-0.1500000E -01	-0.1500000E -01	0.2470024E 04	0.4480103E 02	0.1125114E 03	-0.1500000E -01	-0.1500000E -01	-0.1500000E -01	-0.1500000E -01	-0.1500000E -01
0.2771170E 04	0.1499577E 03	0.1722205E 03	-0.1550000E -01	-0.1550000E -01	-0.1550000E -01	-0.1550000E -01	-0.1550000E -01	0.2771170E 04	0.1499577E 03	0.1722205E 03	-0.1550000E -01	-0.1550000E -01	-0.1550000E -01	-0.1550000E -01	-0.1550000E -01
TENSILE END LOADS								TENSILE END LOADS							
0.1785804E 02	0.1681961E 01	0.1165250E 02	0.1000000E -01	0.1000000E -01	0.1000000E -01	0.1000000E -01	0.1000000E -01	0.1785804E 02	0.1681961E 01	0.1165250E 02	0.1000000E -01	0.1000000E -01	0.1000000E -01	0.1000000E -01	0.1000000E -01
0.1629202E 02	0.1757570E 01	0.1119487E 02	0.4000000E -01	0.4000000E -01	0.4000000E -01	0.4000000E -01	0.4000000E -01	0.1629202E 02	0.1757570E 01	0.1119487E 02	0.4000000E -01	0.4000000E -01	0.4000000E -01	0.4000000E -01	0.4000000E -01
0.1579238E 02	0.1616758E 01	0.1109081E 02	0.5000000E -01	0.5000000E -01	0.5000000E -01	0.5000000E -01	0.5000000E -01	0.1579238E 02	0.1616758E 01	0.1109081E 02	0.5000000E -01	0.5000000E -01	0.5000000E -01	0.5000000E -01	0.5000000E -01
0.1358770E 02	0.1497246E 01	0.1104999E 02	0.6000000E -01	0.6000000E -01	0.6000000E -01	0.6000000E -01	0.6000000E -01	0.1358770E 02	0.1497246E 01	0.1104999E 02	0.6000000E -01	0.6000000E -01	0.6000000E -01	0.6000000E -01	0.6000000E -01
0.1320150E 02	0.1385178E 01	0.1018405E 02	0.7000000E -01	0.7000000E -01	0.7000000E -01	0.7000000E -01	0.7000000E -01	0.1320150E 02	0.1385178E 01	0.1018405E 02	0.7000000E -01	0.7000000E -01	0.7000000E -01	0.7000000E -01	0.7000000E -01
0.1203770E 02	0.1186658E 01	0.1099055E 01	0.8000000E -01	0.8000000E -01	0.8000000E -01	0.8000000E -01	0.8000000E -01	0.1203770E 02	0.1186658E 01	0.1099055E 01	0.8000000E -01	0.8000000E -01	0.8000000E -01	0.8000000E -01	0.8000000E -01
0.1078749E 02	0.1024903E 01	0.1055036E 01	0.9000000E -01	0.9000000E -01	0.9000000E -01	0.9000000E -01	0.9000000E -01	0.1078749E 02	0.1024903E 01	0.1055036E 01	0.9000000E -01	0.9000000E -01	0.9000000E -01	0.9000000E -01	0.9000000E -01
0.9587154E 01	0.1280470E 01	0.1215913E 01	0.1000000E -01	0.1000000E -01	0.1000000E -01	0.1000000E -01	0.1000000E -01	0.9587154E 01	0.1280470E 01	0.1215913E 01	0.1000000E -01	0.1000000E -01	0.1000000E -01	0.1000000E -01	0.1000000E -01
0.8658570E 01	0.1270074E 01	0.1171271E 01	0.1100000E -01	0.1100000E -01	0.1100000E -01	0.1100000E -01	0.1100000E -01	0.8658570E 01	0.1270074E 01	0.1171271E 01	0.1100000E -01	0.1100000E -01	0.1100000E -01	0.1100000E -01	0.1100000E -01
0.7829563E 01	0.1258805E 01	0.1135880E 01	0.1200000E -01	0.1200000E -01	0.1200000E -01	0.1200000E -01	0.1200000E -01	0.7829563E 01	0.1258805E 01	0.1135880E 01	0.1200000E -01	0.1200000E -01	0.1200000E -01	0.1200000E -01	0.1200000E -01
0.6956066E 01	0.1249386E 01	0.1108168E 01	0.1300000E -01	0.1300000E -01	0.1300000E -01	0.1300000E -01	0.1300000E -01	0.6956066E 01	0.1249386E 01	0.1108168E 01	0.1300000E -01	0.1300000E -01	0.1300000E -01	0.1300000E -01	0.1300000E -01
0.5951516E 01	0.1235068E 01	0.1080788E 01	0.1400000E -01	0.1400000E -01	0.1400000E -01	0.1400000E -01	0.1400000E -01	0.5951516E 01	0.1235068E 01	0.1080788E 01	0.1400000E -01	0.1400000E -01	0.1400000E -01	0.1400000E -01	0.1400000E -01
0.4891055E 01	0.1215024E 01	0.1058166E 01	0.1500000E -01	0.1500000E -01	0.1500000E -01	0.1500000E -01	0.1500000E -01	0.4891055E 01	0.1215024E 01	0.1058166E 01	0.1500000E -01	0.1500000E -01	0.1500000E -01	0.1500000E -01	0.1500000E -01
0.4219941E 01	0.1198729E 01	0.1039123E 01	0.1600000E -01	0.1600000E -01	0.1600000E -01	0.1600000E -01	0.1600000E -01	0.4219941E 01	0.1198729E 01	0.1039123E 01	0.1600000E -01	0.1600000E -01	0.1600000E -01	0.1600000E -01	0.1600000E -01
0.3662974E 01	0.1186751E 01	0.1022088E 01	0.1700000E -01	0.1700000E -01	0.1700000E -01	0.1700000E -01	0.1700000E -01	0.3662974E 01	0.1186751E 01	0.1022088E 01	0.1700000E -01	0.1700000E -01	0.1700000E -01	0.1700000E -01	0.1700000E -01
0.3288533E 01	0.1175583E 01	0.1006112E 01	0.1800000E -01	0.1800000E -01	0.1800000E -01	0.1800000E -01	0.1800000E -01	0.3288533E 01	0.1175583E 01	0.1006112E 01	0.1800000E -01	0.1800000E -01	0.1800000E -01	0.1800000E -01	0.1800000E -01
0.2974310E 01	0.1165117E 01	0.0991349E 01	0.1900000E -01	0.1900000E -01	0.1900000E -01	0.1900000E -01	0.1900000E -01	0.2974310E 01	0.1165117E 01	0.0991349E 01	0.1900000E -01	0.1900000E -01	0.1900000E -01	0.1900000E -01	0.1900000E -01
0.2718427E 01	0.1155493E 01	0.0978465E 01	0.2000000E -01	0.2000000E -01	0.2000000E -01	0.2000000E -01	0.2000000E -01	0.2718427E 01	0.1155493E 01	0.0978465E 01	0.2000000E -01	0.2000000E -01	0.2000000E -01	0.2000000E -01	0.2000000E -01
0.2521880E 01	0.1146812E 01	0.0965862E 01	0.2100000E -01	0.2100000E -01	0.2100000E -01	0.2100000E -01	0.2100000E -01	0.2521880E 01	0.1146812E 01	0.0965862E 01	0.2100000E -01	0.2100000E -01	0.2100000E -01	0.2100000E -01	0.2100000E -01
0.1980204E 01	0.1138707E 01	0.0953225E 01	0.2200000E -01	0.2200000E -01	0.2200000E -01	0.2200000E -01	0.2200000E -01	0.1980204E 01	0.1138707E 01	0.0953225E 01	0.2200000E -01	0.2200000E -01	0.22000		

TABLE 1.5-2 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_2 , \bar{T} AND \bar{Q}

[illegible]

TABLE 1.5-2 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_d , \bar{T} AND \bar{Q}

T _D * 0.16000000 C7				Q * 0.60000000 D1				T _D * -0.16000000 D2				Q * 0.70000000 D1			
Y		M		X				Y		M		X			
COMPRESSIVE END LOADS								COMPRESSIVE END LOADS							
0.13013331 03	0.10000000 07	0.60000000 01	0.	0.15613331 03	0.11000000 07	0.90000000 01	0.	0.16812701 03	0.11623188 02	0.10028111 02	-0.10000000 -03	0.17883044 03	0.11775746 02	0.10388121 02	-0.00000000 -03
0.13011155 01	0.10138122 02	0.69547574 01	-0.10000000 -03	0.16812701 03	0.11623188 02	0.10028111 02	-0.10000000 -03	0.19114131 03	0.12261191 02	0.12084151 02	-0.00000000 -03	0.21278811 03	0.12791181 02	0.13643101 02	-0.00000000 -03
0.13009794 01	0.11174781 02	0.88079711 01	-0.00000000 00	0.21278811 03	0.12791181 02	0.13643101 02	-0.00000000 -03	0.24265193 03	0.13773001 02	0.15778771 02	-0.00000000 -03	0.28306803 03	0.14919772 02	0.18548671 02	-0.00000000 -03
0.20021371 03	0.12523502 02	0.12133870 02	-0.00000000 00	0.28306803 03	0.14919772 02	0.18548671 02	-0.00000000 -03	0.34697781 03	0.16476616 02	0.22341191 02	-0.00000000 -03	0.43938181 03	0.18627772 02	0.27627772 02	-0.10000000 01
0.23651521 03	0.13574781 02	0.18607498 02	-0.00000000 00	0.43938181 03	0.18627772 02	0.27627772 02	-0.10000000 01	0.59861781 03	0.21776011 02	0.33438981 02	-0.11000000 01	0.89603311 03	0.26897491 02	0.45848191 02	-0.12000000 01
0.28727691 03	0.14989846 02	0.10114876 02	-0.00000000 00	0.59861781 03	0.21776011 02	0.33438981 02	-0.11000000 01	0.89603311 03	0.26897491 02	0.45848191 02	-0.12000000 01	0.13563291 03	0.13563291 02	0.38973091 02	-0.13000000 01
0.36870171 03	0.16897502 02	0.22997531 02	-0.10000000 01	0.89603311 03	0.26897491 02	0.45848191 02	-0.12000000 01	0.13563291 03	0.13563291 02	0.38973091 02	-0.13000000 01	0.16895781 04	0.16895781 02	0.11563791 01	-0.14000000 01
0.49810941 03	0.19824891 02	0.32993051 02	-0.11000000 01	0.16895781 04	0.16895781 02	0.11563791 01	-0.14000000 01	0.70977791 04	0.70977791 02	0.16791001 01	-0.14500000 01	0.70977791 04	0.70977791 02	0.16791001 01	-0.14500000 01
0.74575811 03	0.24136881 02	0.51070101 02	-0.12000000 01	0.70977791 04	0.70977791 02	0.16791001 01	-0.14500000 01	0.12782721 05	0.12782721 02	0.29677791 03	-0.15000000 01	0.12782721 05	0.12782721 02	0.29677791 03	-0.15000000 01
0.11010701 04	0.12712701 02	0.20938101 02	0.13000000 01	0.12782721 05	0.12782721 02	0.29677791 03	-0.15000000 01	0.22241791 06	0.22241791 02	0.10329791 04	-0.15500000 01	0.22241791 06	0.22241791 02	0.10329791 04	-0.15500000 01
0.05511651 04	0.09881591 02	0.10298731 01	0.14000000 01	0.22241791 06	0.22241791 02	0.10329791 04	-0.15500000 01								
0.57073491 04	0.26902161 02	0.15918671 01	0.14500000 01												
0.10624381 05	0.11582001 03	0.18697501 01	0.15000000 01												
0.10619771 06	0.18895701 03	0.09373731 03	0.15500000 01												
TENSILE END LOADS								TENSILE END LOADS							
0.12118171 01	0.08811231 01	0.01313791 01	0.10000000 01	0.16584451 01	0.10380091 02	0.04985871 01	0.10000000 01	0.13177751 01	0.10314141 02	0.08887771 01	0.10000000 01	0.12144471 01	0.09791181 01	0.05312111 01	0.10000000 01
0.11885441 01	0.09197191 01	0.04929551 01	0.10000000 01	0.13177751 01	0.10314141 02	0.08887771 01	0.10000000 01	0.12144471 01	0.09791181 01	0.05312111 01	0.10000000 01	0.11320611 01	0.09470271 01	0.05525551 01	0.10000000 01
0.10747191 01	0.10612901 01	0.07146001 01	0.10000000 01	0.10314141 02	0.08887771 01	0.08887771 01	0.10000000 01	0.11320611 01	0.09470271 01	0.05525551 01	0.10000000 01	0.10147201 01	0.09184791 01	0.04517731 01	0.10000000 01
0.09273631 02	0.07171781 01	0.02677731 01	0.10000000 01	0.10147201 01	0.09184791 01	0.04517731 01	0.10000000 01	0.09184791 01	0.04517731 01	0.04517731 01	0.10000000 01	0.07881821 02	0.08597491 01	0.04335121 01	0.10000000 01
0.07071851 02	0.04911161 01	0.13277131 01	0.10000000 01	0.07881821 02	0.08597491 01	0.04335121 01	0.10000000 01	0.07881821 02	0.08597491 01	0.04335121 01	0.10000000 01	0.06186101 02	0.08178101 01	0.03278781 01	0.10000000 01
0.02979191 02	0.02247711 01	0.04944791 01	0.10000000 01	0.06186101 02	0.08178101 01	0.03278781 01	0.10000000 01	0.06186101 02	0.08178101 01	0.03278781 01	0.10000000 01	0.04288881 02	0.07176791 01	0.02725191 01	0.10000000 01
0.07627861 02	0.05713401 01	0.05755801 01	0.10000000 01	0.04288881 02	0.07176791 01	0.02725191 01	0.10000000 01	0.04288881 02	0.07176791 01	0.02725191 01	0.10000000 01	0.03788881 02	0.06788881 01	0.02146671 01	0.10000000 01
0.04080511 02	0.03710071 01	0.13081571 01	0.10000000 01	0.03788881 02	0.06788881 01	0.02146671 01	0.10000000 01	0.03788881 02	0.06788881 01	0.02146671 01	0.10000000 01	0.02671271 02	0.04671271 01		
0.03593451 02	0.03000711 01	0.03000711 01	0.10000000 01	0.02671271 02	0.04671271 01			0.02671271 02	0.04671271 01			0.02671271 02	0.04671271 01		
0.02191431 02	0.02000711 01	0.02000711 01	0.10000000 01	0.02000711 01	0.02000711 01			0.02000711 01	0.02000711 01			0.02000711 01	0.02000711 01		
0.13072431 02	0.14889841 01	0.23888881 01	0.10000000 01	0.14889841 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14889841 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14889841 01	0.23888881 01	0.23888881 01	0.10000000 01
0.13060251 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01
0.12710901 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01
0.11300151 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01
0.11300151 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01
0.11300151 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01
0.11300151 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01
0.11300151 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01
0.11300151 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01
0.11300151 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01
0.11300151 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01
0.11300151 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01
0.11300151 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01
0.11300151 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01
0.11300151 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01
0.11300151 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01
0.11300151 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01
0.11300151 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01
0.11300151 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01
0.11300151 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01	0.10000000 01
0.11300151 02	0.14771091 01	0.23888881 01	0.10000000 01	0.14771091 01	0.23888881 01	0.23888881 01</									

[illegible]

TABLE 1.5-2 (Cont'd)
VALUES OF $\bar{\lambda}$, \bar{y} AND \bar{M} FOR SPECIFIED VALUES OF \bar{T}_d , \bar{T} AND \bar{Q}

[illegible]

TABLE 1.5-2 (Cont'd)
VALUES OF \bar{Y} , \bar{M} AND \bar{X} FOR SPECIFIED VALUES OF \bar{T}_d , \bar{T} AND \bar{Q}

$\bar{T}_d = -0.2000000E 02$				$\bar{T}_d = -0.2000000E 02$			
\bar{T}	\bar{Y}	\bar{M}	\bar{X}	\bar{T}	\bar{Y}	\bar{M}	\bar{X}
COMPRESSIVE END LOADS				COMPRESSIVE END LOADS			
0.2525337E 03	0.1400000E 02	0.1203000E 02	0.	0.2893531E 03	0.1500000E 02	0.1500000E 02	0.
0.2711020E 03	0.1453889E 02	0.1316000E 02	-0.3000000E -03	0.3106877E 03	0.1557603E 02	0.1640184E 02	-0.3000000E -03
0.2885900E 03	0.1498876E 02	0.1449700E 02	-0.4000000E -04	0.3295380E 03	0.1605521E 02	0.1756863E 02	-0.4000000E -04
0.3173850E 03	0.1568455E 02	0.1590114E 02	-0.5000000E -05	0.3567209E 03	0.1671581E 02	0.1917895E 02	-0.5000000E -05
0.3457035E 03	0.1646113E 02	0.1771510E 02	-0.6000000E -06	0.3948041E 03	0.1759796E 02	0.2135509E 02	-0.6000000E -06
0.3924452E 03	0.1752758E 02	0.2058811E 02	-0.7000000E -07	0.4482451E 03	0.1877180E 02	0.2419822E 02	-0.7000000E -07
0.4492894E 03	0.1898401E 02	0.2415106E 02	-0.8000000E -08	0.5246174E 03	0.2045157E 02	0.2801271E 02	-0.8000000E -08
0.5360113E 03	0.2095440E 02	0.2897663E 02	-0.9000000E -09	0.6374458E 03	0.2244409E 02	0.3317715E 02	-0.9000000E -09
0.7114286E 03	0.2370521E 02	0.3570521E 02	-0.1000000E -10	0.8127878E 03	0.2537743E 02	0.4037743E 02	-0.1000000E -10
0.9682741E 03	0.2770750E 02	0.4552585E 02	-0.1100000E -11	0.1106367E 04	0.2965642E 02	0.5083426E 02	-0.1100000E -11
0.1469467E 04	0.3396716E 02	0.6491558E 02	-0.1200000E -12	0.1556327E 04	0.3635136E 02	0.6734596E 02	-0.1200000E -12
0.2550202E 04	0.4498019E 02	0.8801684E 02	-0.1300000E -13	0.2891560E 04	0.512391E 02	0.9632940E 02	-0.1300000E -13
0.5936666E 04	0.6906425E 02	0.1473659E 03	-0.1400000E -14	0.6785062E 04	0.87244E 02	0.1597900E 03	-0.1400000E -14
0.1447687E 05	0.9814677E 02	0.2141486E 03	-0.1500000E -15	0.1311728E 05	0.1028266E 03	0.2511929E 03	-0.1500000E -15
0.3252422E 05	0.1315769E 03	0.3754400E 03	-0.1500000E -15	0.3645100E 05	0.1727782E 03	0.4037509E 03	-0.1500000E -15
0.5627429E 06	0.1819547E 03	0.113770E 04	-0.1550000E -15	0.4145934E 06	0.2794196E 03	0.1407053E 04	-0.1550000E -15
TENSILE END LOADS				TENSILE END LOADS			
0.2351542E 03	0.1347940E 02	0.1078506E 02	0.3000000E -09	0.2647361E 03	0.1446457E 02	0.1369819E 02	0.3000000E -09
0.2278527E 03	0.1313164E 02	0.9396616E 01	0.4000000E -10	0.2546129E 03	0.1407153E 02	0.1274823E 02	0.4000000E -10
0.2054092E 03	0.1269174E 02	0.8877139E 01	0.5000000E -11	0.2379128E 03	0.1360043E 02	0.1159389E 02	0.5000000E -11
0.1925917E 03	0.1218897E 02	0.7612005E 01	0.6000000E -12	0.2198390E 03	0.1306119E 02	0.1029775E 02	0.6000000E -12
0.1761719E 03	0.1164147E 02	0.6296730E 01	0.7000000E -13	0.2010207E 03	0.1247990E 02	0.8984847E 01	0.7000000E -13
0.1596220E 03	0.1107102E 02	0.4914550E 01	0.8000000E -14	0.1821647E 03	0.1186768E 02	0.7404688E 01	0.8000000E -14
0.1435856E 03	0.1048576E 02	0.3506551E 01	0.9000000E -15	0.1638475E 03	0.1124174E 02	0.5894191E 01	0.9000000E -15
0.1283720E 03	0.9897795E 01	0.2400751E 01	0.1000000E -16	0.1484675E 03	0.1061500E 02	0.4384990E 01	0.1000000E -16
0.1147154E 03	0.9422812E 01	0.1719777E 00	0.1100000E -17	0.1352984E 03	0.9997801E 01	0.2902577E 01	0.1100000E -17
0.1012443E 03	0.8767111E 01	0.1177350E 00	0.1200000E -18	0.1154811E 03	0.9198379E 01	0.1864436E 01	0.1200000E -18
0.8950430E 02	0.8273731E 01	0.1990057E 01	0.1300000E -19	0.1020737E 03	0.8822169E 01	0.9053450E -01	0.1300000E -19
0.7898036E 02	0.7710674E 01	0.3117331E 01	0.1400000E -20	0.9005689E 02	0.8273720E 01	0.1215649E 01	0.1400000E -20
0.6961653E 02	0.7225786E 01	0.4756895E 01	0.1500000E -21	0.7956649E 02	0.7754041E 01	0.2446597E 01	0.1500000E -21
0.6133079E 02	0.6768720E 01	0.5327922E 01	0.1600000E -22	0.6990950E 02	0.7265544E 01	0.3599793E 01	0.1600000E -22
0.5402940E 02	0.6341074E 01	0.6525765E 01	0.1700000E -23	0.6157794E 02	0.68071E 01	0.4675041E 01	0.1700000E -23
0.4761518E 02	0.5941233E 01	0.7751867E 01	0.1800000E -24	0.5425809E 02	0.6380017E 01	0.5673852E 01	0.1800000E -24
0.4126454E 02	0.5570297E 01	0.8108771E 01	0.1900000E -25	0.4784020E 02	0.5981081E 01	0.6598922E 01	0.1900000E -25
0.3705093E 02	0.5224947E 01	0.8499770E 01	0.2000000E -26	0.4221665E 02	0.5613417E 01	0.7453749E 01	0.2000000E -26
0.3272674E 02	0.4904450E 01	0.8678670E 01	0.2100000E -27	0.3729038E 02	0.5270365E 01	0.8242110E 01	0.2100000E -27
0.2893513E 02	0.4607111E 01	0.1079959E 01	0.2200000E -28	0.3297289E 02	0.4952718E 01	0.8948834E 01	0.2200000E -28
0.2560650E 02	0.4331349E 01	0.1071601E 01	0.2300000E -29	0.2918524E 02	0.4657396E 01	0.9637621E 01	0.2300000E -29
0.2260000E 02	0.4076829E 01	0.1144253E 02	0.2400000E -30	0.2565709E 02	0.4384120E 01	0.1025222E 02	0.2400000E -30
0.2010216E 02	0.3894052E 01	0.1200283E 02	0.2500000E -31	0.2292904E 02	0.4131022E 01	0.1091449E 02	0.2500000E -31
0.1732457E 02	0.3671593E 01	0.1244061E 02	0.2600000E -32	0.2034608E 02	0.3894165E 01	0.1135941E 02	0.2600000E -32
0.1501218E 02	0.3418797E 01	0.1271932E 02	0.2700000E -33	0.1806250E 02	0.3657757E 01	0.1131427E 02	0.2700000E -33
0.1402457E 02	0.3229501E 01	0.1322424E 02	0.2800000E -34	0.1638297E 02	0.3478899E 01	0.1225809E 02	0.2800000E -34
0.1243337E 02	0.3055025E 01	0.1369276E 02	0.2900000E -35	0.1473488E 02	0.3289440E 01	0.1266653E 02	0.2900000E -35
0.1101234E 02	0.2897590E 01	0.1403372E 02	0.3000000E -36	0.1263444E 02	0.3115157E 01	0.1304317E 02	0.3000000E -36

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1.5 (Cont'd)

CASE C - Transverse load uniformly distributed or concentrated at midspan - specified axial loads (zero axial end restraint):

The beam is shown schematically in Figure 1.5-3. Since the ends are unrestrained axially, the beam ends are free to move due to the action of temperature, transverse loads and specified end loads. Hence the compatibility Eqs. (2) and (5) do not apply, and since λ is now a known quantity, the maximum deflections and bending moments may be determined directly from Eqs. (3) and (6). These quantities are independent of the average temperature in the beam.

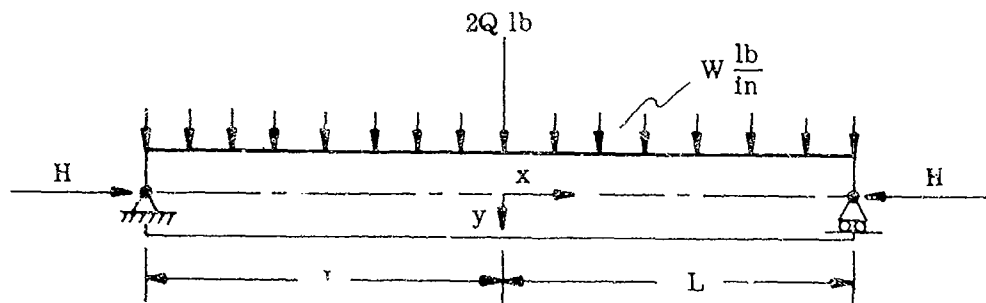
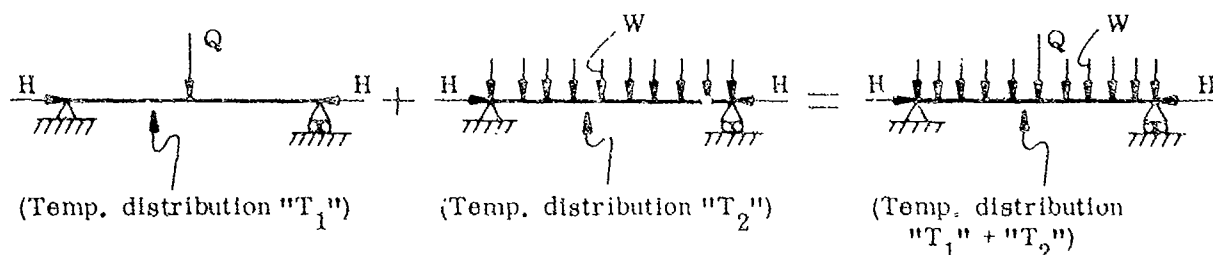
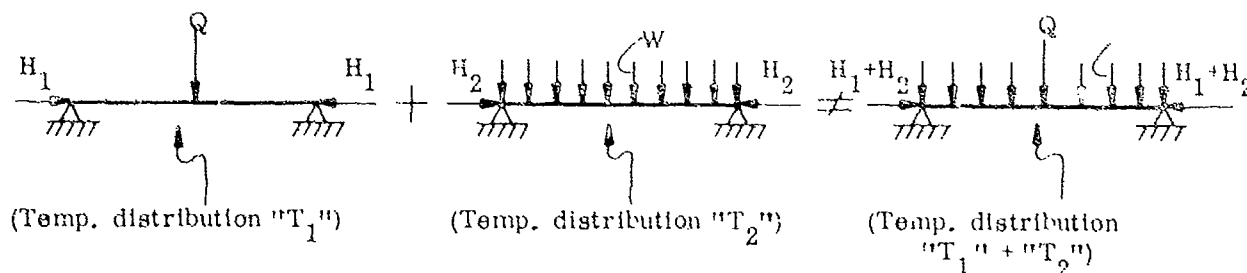


FIGURE 1.5-3. RECTANGULAR BEAM WITH ZERO AXIAL END RESTRAINT AND SPECIFIED AXIAL LOADS

It is important to note that when the end loadings are specified then a modified principle of superposition as shown by Figure 1.5-4a may be used. The figure shows that the resultant effect of several transverse loads and temperature distributions acting simultaneously



(a) Superposition valid for an axially unrestrained beam with specified end loads " H "



(b) Superposition not valid for an axially restrained beam

FIGURE 1.5-4 MODIFIED SUPERPOSITION PRINCIPLE

1.5 (Cont'd)

in the presence of a specified axial load can be obtained by superposing the effects of the individual transverse loads and temperatures acting with the axial load.

However, this superposition principle is not valid when the ends are restrained axially (e.g., Cases A and B). In such cases the axial loads depend on the transverse loads and temperatures nonlinearly.

1.6 USE OF THE TABLES

Tables 1.5-1 and 1.5-2 are reproductions of IBM 7090 digital computer numerical solutions for the beam-column problem. Table 1.5-1 presents results for the case of uniform transverse loading and Table 1.5-2 applies for concentrated midspan loads. Each table is first subdivided into sections corresponding to given temperature differences and transverse loads and then further subdivided into compressive and tensile end loading cases. Since the quantities \bar{W} , \bar{T}_d , and \bar{T} specified in a given problem will not in general coincide with those listed in the tables, interpolation must be employed. The important quantity to be evaluated is the nondimensional end loading parameter, $\bar{\lambda}$ and the spacing of the tabulated values has been made sufficiently close so as to allow reasonably accurate interpolation for engineering purposes. A systematic interpolation formula will be given as part of an illustrative problem in Sub-section 1.7.

Numerical values are given in terms of a floating decimal number system and are to be interpreted as shown by the following examples

$$0.2456582E\ 00 = 0.2456582 \times 10^0 = 0.2456582$$

$$0.2456582E\ 02 = 0.2456582 \times 10^2 = 24.56582$$

$$0.2456582E-01 = 0.2456582 \times 10^{-1} = 0.02456582$$

1.7 NUMERICAL EXAMPLES

EXAMPLE I - Beam with Full Axial End Restraint:

Figure 1.7-1 shows a simply supported strip with immovable ends subjected to a uniformly distributed load of $2 \frac{\text{lb}}{\text{in}}$. The temperature varies linearly through the thickness from 100° F at the upper face to 150° F at the lower face. Young's modulus and the linear coefficient of thermal expansion are taken to be

$$E = 30 \times 10^6 \text{ psi}$$

$$\alpha = 6 \times 10^{-6} \text{ in/in} - ^\circ \text{F}.$$

Find the axial end load, midspan deflection, bending moment and the maximum stress.

1.7 (Cont'd)

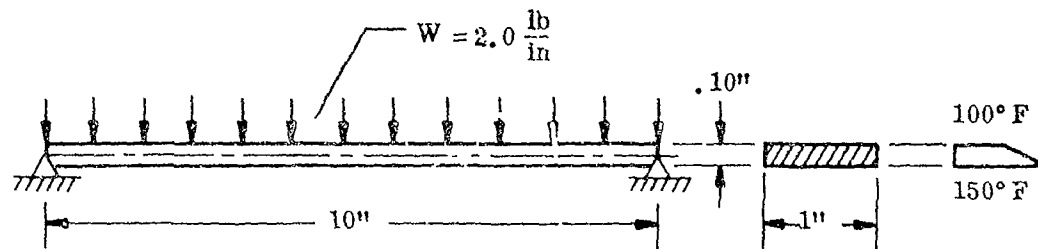


FIGURE 1.7-1 ILLUSTRATIVE PROBLEM; STRIP WITH IMMOVABLE ENDS

SOLUTION

From Figure 1.7-1 and the given data:

$$\begin{aligned} b &= 1'' & T_o &= 100^\circ \text{F} \\ h &= 0.10'' & T_i &= 150^\circ \text{F} \\ L &= 5'' & E &= 30 \times 10^6 \text{ psi} \\ W &= 2.0 \frac{\text{lb}}{\text{in}} & \alpha &= 6 \times 10^{-6} \text{ in/in} - ^\circ \text{F} \end{aligned}$$

Therefore

$$\bar{W} = \frac{12W}{Eb} \left(\frac{L}{h} \right)^4 = \frac{12(2.0)}{(30)(10^6)(1)} \left(\frac{5}{0.10} \right)^4 = 5.0$$

$$\bar{T}_d = \alpha \left(\frac{L}{h} \right)^2 (T_o - T_i) = 6(10)^{-6} \left(\frac{5}{0.10} \right)^2 (100 - 150) = -0.75$$

$$\bar{T} = \alpha \left(\frac{L}{h} \right)^2 (T_o + T_i) = 6(10)^{-6} \left(\frac{5}{0.10} \right)^2 (100 + 150) = 3.75$$

Using the above nondimensional quantities we must now utilize Table 1.5-1 to determine the nondimensional axial loading parameter $\bar{\lambda}$. Since the table does not give $\bar{\lambda}$ directly for the above combination of quantities the following interpolation procedure is recommended:

- (1) Determine the next lower and higher values of \bar{W} that are given in the table. Designate these values as \bar{W}_0 and \bar{W}_1 respectively. For this example

$$\bar{W}_0 = 3.0 \text{ and}$$

$$\bar{W}_1 = 6.0 .$$

1.7 (Cont'd)

- (2) Determine the next lower and higher values of \bar{T}_d that are given in the table. Designate these values as \bar{T}_{d0} and \bar{T}_{d1} , respectively.

For this example

$$\bar{T}_{d0} = -4.0 \quad \text{and}$$

$$\bar{T}_{d1} = 0.$$

- (3) For each of the four combinations $(\bar{W}_0, \bar{T}_{d0})$, $(\bar{W}_0, \bar{T}_{d1})$, $(\bar{W}_1, \bar{T}_{d0})$, $(\bar{W}_1, \bar{T}_{d1})$ the table lists $\bar{\lambda}$'s corresponding to \bar{T} 's on either side of the given value for \bar{T} . Denote the values of \bar{T} listed in the tables for the combination $(\bar{W}_1, \bar{T}_{dj})$ by $(\bar{T}_{ij}, \bar{T}'_{ij})$ and the corresponding values of $\bar{\lambda}$ by $(\bar{\lambda}_{ij}, \bar{\lambda}'_{ij})$ where $\bar{T}_{ij} < \bar{T} < \bar{T}'_{ij}$

For this example

$$\bar{T}_{00} = 3.397$$

$$\bar{\lambda}_{00} = 1.2$$

$$\bar{T}'_{00} = 3.896$$

$$\bar{\lambda}'_{00} = 1.1$$

$$\bar{T}_{01} = 3.041$$

$$\bar{\lambda}_{01} = -1.2$$

$$\bar{T}'_{01} = 5.174$$

$$\bar{\lambda}'_{01} = -1.3$$

$$\bar{T}_{10} = 3.416$$

$$\bar{\lambda}_{10} = 1.5$$

$$\bar{T}'_{10} = 3.969$$

$$\bar{\lambda}'_{10} = 1.4$$

$$\bar{T}_{11} = 3.648$$

$$\bar{\lambda}_{11} = -.8$$

$$\bar{T}'_{11} = 4.441$$

$$\bar{\lambda}'_{11} = -.9$$

- (4) The value of $\bar{\lambda}$ may now be obtained from the following interpolation formula.

$$\bar{\lambda} = \left[\frac{\bar{W} - \bar{W}_0}{\bar{W}_1 - \bar{W}_0} \right] A + \left[\frac{\bar{W}_1 - \bar{W}}{\bar{W}_1 - \bar{W}_0} \right] B$$

1.7 (Cont'd)

where

$$A = \left[\frac{\bar{T}_d - \bar{T}_{d0}}{\bar{T}_{d1} - \bar{T}_{d0}} \right] \left[\frac{(\bar{T} - \bar{T}_{11}) \bar{\lambda}'_{11} + (\bar{T}'_{11} - \bar{T}) \bar{\lambda}_{11}}{\bar{T}'_{11} - \bar{T}_{11}} \right]$$

$$+ \left[\frac{\bar{T}_{d1} - \bar{T}_d}{\bar{T}_{d1} - \bar{T}_{d0}} \right] \left[\frac{(\bar{T} - \bar{T}_{10}) \bar{\lambda}'_{10} + (\bar{T}'_{10} - \bar{T}) \bar{\lambda}_{10}}{\bar{T}'_{10} - \bar{T}_{10}} \right]$$

$$B = \left[\frac{\bar{T}_d - \bar{T}_{d0}}{\bar{T}_{d1} - \bar{T}_{d0}} \right] \left[\frac{(\bar{T} - \bar{T}_{01}) \bar{\lambda}'_{01} + (\bar{T}'_{01} - \bar{T}) \bar{\lambda}_{01}}{\bar{T}'_{01} - \bar{T}_{01}} \right]$$

$$+ \left[\frac{\bar{T}_{d1} - \bar{T}_d}{\bar{T}_{d1} - \bar{T}_{d0}} \right] \left[\frac{(\bar{T} - \bar{T}_{00}) \bar{\lambda}'_{00} + (\bar{T}'_{00} - \bar{T}) \bar{\lambda}_{00}}{\bar{T}'_{00} - \bar{T}_{00}} \right]$$

Substituting the known quantities into the above formula yields

$$\bar{\lambda} = -.524$$

Thus the axial end load is given by

$$H = \frac{E\bar{\lambda}^2}{L^2} = \frac{(30)(10)^6 [(1)(.10)^{3/12}] (-.524)^2}{(5)^2} = 27.4 \text{ lb. (compression)}$$

The nondimensional deflection and bending moment at midspan can be determined from Table 1.5-1 using an identical interpolation procedure. However, a simpler and more accurate method is to substitute the known values of $\bar{\lambda}$, \bar{W} , and \bar{T}_d into Eqs. (3a) of Subsection 1.5*). Direct substitution yields

$$\bar{y} = \left[\frac{y}{h} \right]_{x=0} = 1.60 \text{ and}$$

$$\bar{M} = \left[\frac{12M L^2}{Ebh^4} \right]_{x=0} = 2.94 ,$$

so that

$$y \Big|_{x=0} = h \bar{y} = (.10)(1.60) = .16 \text{ in. (downward)}$$

$$M \Big|_{x=0} = \frac{Ebh^4}{12L^2} \bar{M}$$

$$= \frac{30(10)^6 (.1)^4 (2.94)}{12(5)^2} \quad (2.94)$$

$$= 29.4 \text{ in.-lb (compression in upper fiber)}$$

* Eqs. (3b) are not to be used here since they apply only for $\bar{\lambda} > 0$ (tension)

1.7 (Cont'd)

The maximum stress is thus

$$\sigma_{\max} = \frac{H}{bh} + \frac{M(6)}{bh^2} = 17,900 \text{ psi.}$$

The effect of axially restraining the beam is evidenced by comparing these results with those for an axially unrestrained and simply supported beam subjected to the same transverse loading and temperature but with $H = 0$. In such a case the central deflection, bending moment and maximum stress are given by (Section 4 of Reference 1-2):

$$\begin{aligned} y \Big|_{x=0} &= \frac{5W(2L)^4}{384EI} + \frac{\alpha (T_i - T_o) L^2}{2h} \\ &= .104 + .036 \\ &= .142 \text{ in} \end{aligned}$$

$$M \Big|_{x=0} = \frac{W(2L)^2}{8} = 25 \text{ in-lb.}$$

$$\sigma_{\max} = 15,000 \text{ psi.}$$

Thus, in this case, neglecting axial end restraint which may be present yields unconservative results for both the central deflection and maximum stress.

EXAMPLE II - Unrestrained Beam Column With Prescribed Axial Load:

Figure 1.7-2 shows a simply supported strip with movable ends and a prescribed 20 lb tensile load subjected to a concentrated midspan load of 10 lbs. The temperature varies linearly through the thickness from 200° F at the upper face to 150° F at the bottom face. Young's modulus and the linear coefficient of thermal expansion are given as

$$E = 30 \times 10^6 \text{ psi}$$

$$\alpha = 6 \times 10^{-6} \text{ in./in.} - ^\circ \text{F}$$

Find the midspan deflection, bending moment and the maximum stress.

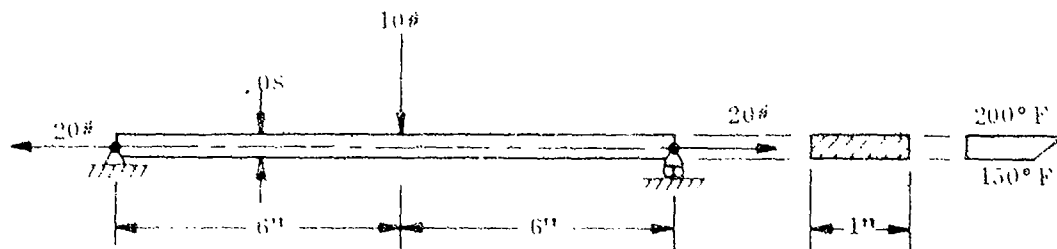


FIGURE 1.7-2 ILLUSTRATIVE PROBLEM; STRIP WITH MOVABLE ENDS

1.7 (Cont'd)

SOLUTION

From Figure 1.7-2 and the given data:

$$b = 1''$$

$$T_o = 200^\circ \text{F}$$

$$h = .08''$$

$$T_i = 150^\circ \text{F}$$

$$L = 6''$$

$$Q = 5 \text{ lb}$$

$$E = 30 \times 10^6 \text{ psi}$$

$$H = 20 \text{ lb}$$

$$\alpha = 6 \times 10^{-6} \text{ in/in} - ^\circ \text{F}$$

Therefore

$$\bar{Q} = \frac{12QL^3}{Ebh^4} = \frac{(12)(5)(6)^3}{30(10)^6(1)(.08)^4} = 10.55$$

$$\bar{T}_d = \alpha \left(\frac{L}{h} \right)^2 (T_o - T_i) = 6(10)^{-6} \left(\frac{6}{.08} \right)^2 (200 - 150) = 1.69$$

$$\bar{\lambda} = \sqrt{\frac{HL^2}{EI}} = \sqrt{\frac{20(6)^2}{(30)(10)^6(1)(.08)^3/12}} = .75$$

As discussed in Sub-section 1.5, since the ends are free to move, the bending response of the beam is independent of the average temperature. The central deflection* and bending moment can thus be obtained by direct substitution of the nondimensional parameters into Eqs. (6b) of Sub-section 1.5. This yields

$$\bar{y} = \left[\frac{y}{h} \right]_{x=0} = .674$$

$$M = \left[\frac{12ML^2}{Ebh^4} \right]_{x=0} = 2.94$$

so that

$$y \Big|_{x=0} = .054 \text{ in (downward)}$$

$$M \Big|_{x=0} = 26.5 \text{ in-lb (compression in upper fiber).}$$

* This is not always the maximum deflection (see Sub-section 1.8)

1.8 ADDITIONAL CONSIDERATIONS

(1) Application of Basic Equations to the Case of Unequal Elastic End Restraints.

The general formulas presented in Sub-section 1.3 have been developed for the case of equal elastic end restraints of stiffness $2K$ (Figure 1.3-1). As is frequently the case, these restraints may be unequal. This situation can be accommodated by replacing the quantity $2K$

in Sub-section 1.3 with the equivalent quantity $\frac{4K_1 K_2}{K_1 + K_2}$ where $2K_1$ and $2K_2$ are the unequal spring stiffnesses. Thus for example in the special case where one end is held ($K_1 = \infty$) then

$\lim_{K_1 \rightarrow \infty} \frac{4K_1 K_2}{K_1 + K_2} = 4K_2$; and this quantity is to be substituted for $2K$ in Eqs. (3) and (4) of

Sub-section 1.4. This procedure will yield the correct results for the deflection and bending moment.

(2) Maximum Bending Moments and Deflections.

This report presents results for the determination of the central deflection and bending moment for a symmetrically loaded beam-column. It can be shown by the usual maximum-minimum procedure that the central bending moment is always numerically the largest bending moment in the beam. If the temperatures and loads tend to produce curvatures in the same direction then the central deflection is a maximum. However where temperature and loads tend to relieve each other, the central deflection may not be a maximum. If this quantity is desired it may be found by considering the full deflection formula (Eq. (1) of Sub-section 1.4).

1.9 REFERENCES

- 1-1 Timoshenko, S., "Theory of Elastic Stability," McGraw Hill Book Company, Inc. pp. 6-8, 1936.
- 1-2 Switzky, H., Forray, M., and Newman, M., "Thermo-Structural Analysis Manual"-Volume I, Sections 1, 2, and 4 of Republic Aviation Corporation Report No. RAC 679-1, September 1960, revised November 1961 (to be published as WADD TR 60-517, Vol. I).

SECTION 2

APPROXIMATE SOLUTION FOR AN AXIALLY RESTRAINED COLUMN
SUBJECTED TO ELEVATED TEMPERATURES AND LATERAL LOADS

by

H. Switzky

SECTION 2

APPROXIMATE SOLUTION FOR AN AXIALLY RESTRAINED COLUMN SUBJECTED TO ELEVATED TEMPERATURES AND LATERAL LOADS

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SECTION 2

APPROXIMATE SOLUTION FOR AN AXIALLY RESTRAINED COLUMN SUBJECTED TO ELEVATED TEMPERATURES AND LATERAL LOADS

2.1 SUMMARY

Approximate solutions are obtained for the axial compressive load and the lateral deflections of columns which are restrained by an axial spring and subjected to an increase in temperature. The solutions are presented graphically in terms of nondimensional parameters.

The problem considered is a column of arbitrary cross section with pinned or clamped ends subjected to an arbitrary temperature and lateral load distribution in addition to finite initial eccentricities. The effects of the thermal gradients, lateral loads, non-linear axial springs, and plasticity of the material are discussed. The simpler problem of constant bending stiffness is explored to illustrate the evaluation of the nondimensional parameters.

2.1.1 Definition of Symbols

The following symbols are used throughout this section:

a	Amplitude of lateral deflection, inches
b	Axial extension parameter ($b = \alpha T / \epsilon_1$)
b	Width of rectangular cross section, inches
d	Axial shortening parameter $\left(d_i = \frac{\Delta_i}{\epsilon_1} \right)$
h	Depth of cross section of column, inches
i	Integer indicating order of the deformation mode
k	Axial stiffness of end restraint, pounds per inch
l	Length of column, inches
m_j	Coefficients expressing the lateral deformation as a polynomial, inches
q	Lateral load, pounds per inch
r	Force function. Ratio of axial load in column to a reference load ($r(x) = F(x) / F_0$)
w	Lateral deflection of column, inches
x	Axial coordinate of column, inches
z	Lateral coordinate of column cross section, inches
α	Coefficient of thermal (linear) expansion, inches per inch $^{\circ}F$
Δ	Deflection of ends of column, inches
Δ	Incremental change
ϵ	Axial strain due to axial load or temperature, inches per inch
ϵ_i	Buckling strain corresponding to i^{th} mode $\left(\epsilon_i = \lambda_i \sigma^2 \right)$
n	Load parameter $\left(n_i = \frac{F}{F_i} \right)$
κ	Curvature of column, 1/inches
λ_i	Eigenvalue for which non-trivial solutions of the differential equilibrium equation exist $\left(\lambda_i = \frac{F_i}{EI} \right)$, 1/sq. inches

2.1.1 (Cont'd)

μ_i	Eigenvalue ($\mu_i = \tan \mu_i$)
ξ	Nondimensional axial coordinate ($\xi = x/l$)
ρ	Effective radius of gyration $= \sqrt{\overline{EI}/\overline{EA}}$, inches
σ	Axial stress ($\sigma = F/A$), psi
C	Stiffness function. Ratio of bending stiffness of cross section to a reference bending stiffness $\left(C(x) = \frac{EI(x)}{E_o I_o} \right)$
ϕ_i	Additional axial shortening function $\left(\phi_i = \left(\frac{1}{1 - \eta_i} \right)^2 - 1 \right)$
Φ	Nondimensional linear axial shortening term $\left(\Phi = 1 + \frac{EA}{kl} + 2 \sum \frac{\eta_i^d}{\eta_i} \right)$
A	Cross sectional area, sq. inches
C	Amplitude of curvature ($v = \sum C_i x_i$)
E_s	Secant modulus ($E_s = \sigma/\epsilon$ for a linear material), psi
\overline{EA}	Axial stiffness ($\overline{EA} = \int E_s dA$, note $\int E_s z dA = 0$), lb
\overline{EI}	Bending stiffness ($\overline{EI} = \int E_s z^2 dA$), pounds square inches
F	Axial load in column, lb
I	Moment of inertia of cross sections, inches ⁴
l	Half length of column ($l = l/2$), inches
M	Moment ($M = EI w''$), pounds per inch
P	Redundant transverse load, lb
T	Temperature increment from datum, °F

SUBSCRIPTS

o	Condition before application of axial load
i	i^{th} mode
q	Due to lateral mechanical loads
T	Due to temperature
$s, s.$	Simple-simple boundary conditions
$s, c.$	Simple-clamped boundary conditions
$c, c.$	Clamped-clamped boundary conditions
s	Symmetrical mode
a	Anti-symmetrical mode

2.1.2 GENERAL APPROACH

The analysis of the problem is approached by examining the deformation characteristics of the column with a linear material to obtain a solution which satisfies compatibility and equilibrium. The following steps are employed to solve the column shown in Figure 2.1.1-1.

- (1) The column deforms in characteristic modes which are dependent upon the distribution of the bending stiffness and axial load in the column as well as the boundary (end) con-

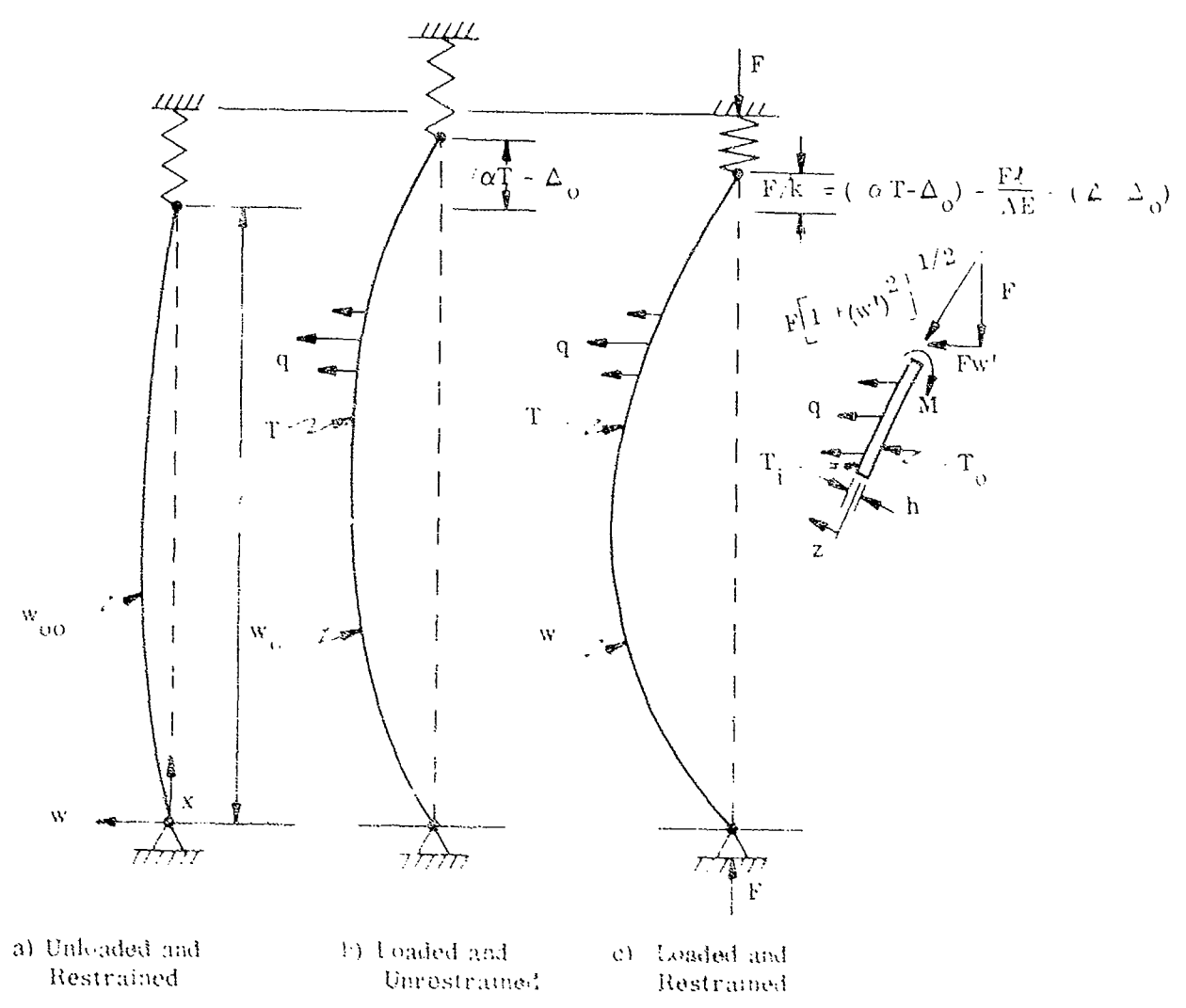


FIGURE 2.1.2-1 RESTRAINED COLUMN

2.1.2 (Cont'd)

ditions. Basic orthogonality relationships exist among the characteristic deformation modes which are used in obtaining a solution for the column in a thermal and mechanical environment.

(2) The deformations of a column subjected to lateral loadings and temperature gradients in addition to an axial load are then determined by solving the equilibrium equation. The analysis results in a solution in terms of the "initial" deformations caused by the lateral loads, thermal gradients, and manufacturing and loading eccentricities acting on a column with no axial load. These "initial" deformations grow with the magnitude of the axial load. Each characteristic deformation mode is magnified as a rectangular hyperbola, by a factor $\left(\frac{1}{1 - r_i} \right)$ which is a function of the applied load and the characteristic "buckling" load corresponding to the deformation mode. The method of determining deformation modes and corresponding buckling loads is illustrated for columns of constant bending stiffness with pinned or clamped ends.

(3) The axial load is then expressed as the solution of a compatibility equation which considers the thermal expansion, the lateral deflection, and the axial deformation of a restraining spring. The axial load determined by this method satisfies both the equilibrium and compatibility conditions of the structure. The solution is, therefore, correct for a reversible (one to one relationship of load and deformation) structure even though a column deforms non-linearly with load.

(4) The compatibility equation is quite difficult to solve but a simple and reasonable approximation of the additional axial shortening of the column due to the axial load reduces the compatibility equation to a form which can be readily solved by graphical means.

(5) The solutions of the axial load and deformation of columns are presented for pinned (simple) or clamped ends simplified by applicable formulae and graphs together with a computational procedure and illustrative problems. The techniques to obtain the "initial" deformations due to lateral load and thermal gradients, and the effective stiffness of the restraining spring are also presented. The effects of nonlinearity in the spring or material are discussed.

2.2 ANALYSIS

The mathematical relationships necessary to solve the problem are derived in Reference 2-1. The results are summarized below.

2.2.1 Relationships of Deformation Modes

Various orthogonality relationships exist between the characteristic deflection modes and their derivatives whenever the boundary conditions are natural (e.g., free, clamped, simple, etc.). These relationships are useful in evaluating the initial and final lateral deformations as well as the axial shortening in terms of these characteristic modes.

Orthogonality relationships, (see Section 1A of Reference 2-1) are obtained by solving the homogeneous differential equilibrium equation

$$(\phi w'')'' + \lambda (rw')' = 0 \quad (1)$$

2.2.1 (Cont'd)

where

- ϕ is the ratio of bending stiffness to a reference bending stiffness ($E_0 I_0$)
- w is the lateral deflection
- r is the ratio of the axial load in the cross section to the reference load (F_0)
- λ is the eigenvalue $\frac{F_0}{E_0 I_0}$.

The following orthogonality relationships are derived

$$\int_0^l (r w_i')' w_k dx = 0 \quad i \neq k \quad (2)$$

Thus the characteristic modes, which are the solutions to the differential equilibrium equation, are orthogonal to the derivative of the weighted slope $[(r w')']$. For end loads ($r = 1$), the deflection and curvature modes are orthogonal. This relationship permits the determination of the amplitudes of the deflection modes for the lateral deflections caused by the lateral load and thermal gradients and permits the determination of the growth of these modes when the column is compressed.

$$\int_0^l r w_i' w_k dx = 0 \quad i \neq k \quad (3)$$

Similarly, the slopes of the deflection modes are orthogonal with respect to a load weighting factor (r). This relationship permits the rapid determination of the axial shortening of the column in terms of the magnitude of the deformation modes.

$$\int_0^l \phi w_i'' w_k'' dx = 0 \quad i \neq k \quad (4)$$

The orthogonality of the curvature with respect to the stiffness weighting function (ϕ), permits the solution of the non-homogeneous differential equilibrium equation resulting from the initial deformations due to the lateral load and thermal gradients. The effect of the lateral load and temperature can be expressed as initial curvatures which grow as a rectangular hyperbola with an increase in axial load as indicated in Figure 2.2.1-1.

2.2.2 Solution of Non-Homogeneous Differential Equilibrium Equation

The homogeneous differential equilibrium equation can be rewritten as a function of the curvature ($\chi = w''$)

$$\text{i.e.} \quad (\phi \chi)'' + \lambda r \chi = 0 \quad (1)$$

The solutions of this equation results in eigenvectors of the curvature which are orthogonal with respect to the weighting function ϕ (Eq. (4) of Paragraph 2.2.1). It is assumed that these curvature modes form a closed set for the "natural" boundary conditions (columns of constant EI ($r = 1$) result in Fourier expansions). Thus, any curvature can be expressed as a weighted sum of the eigenvectors.

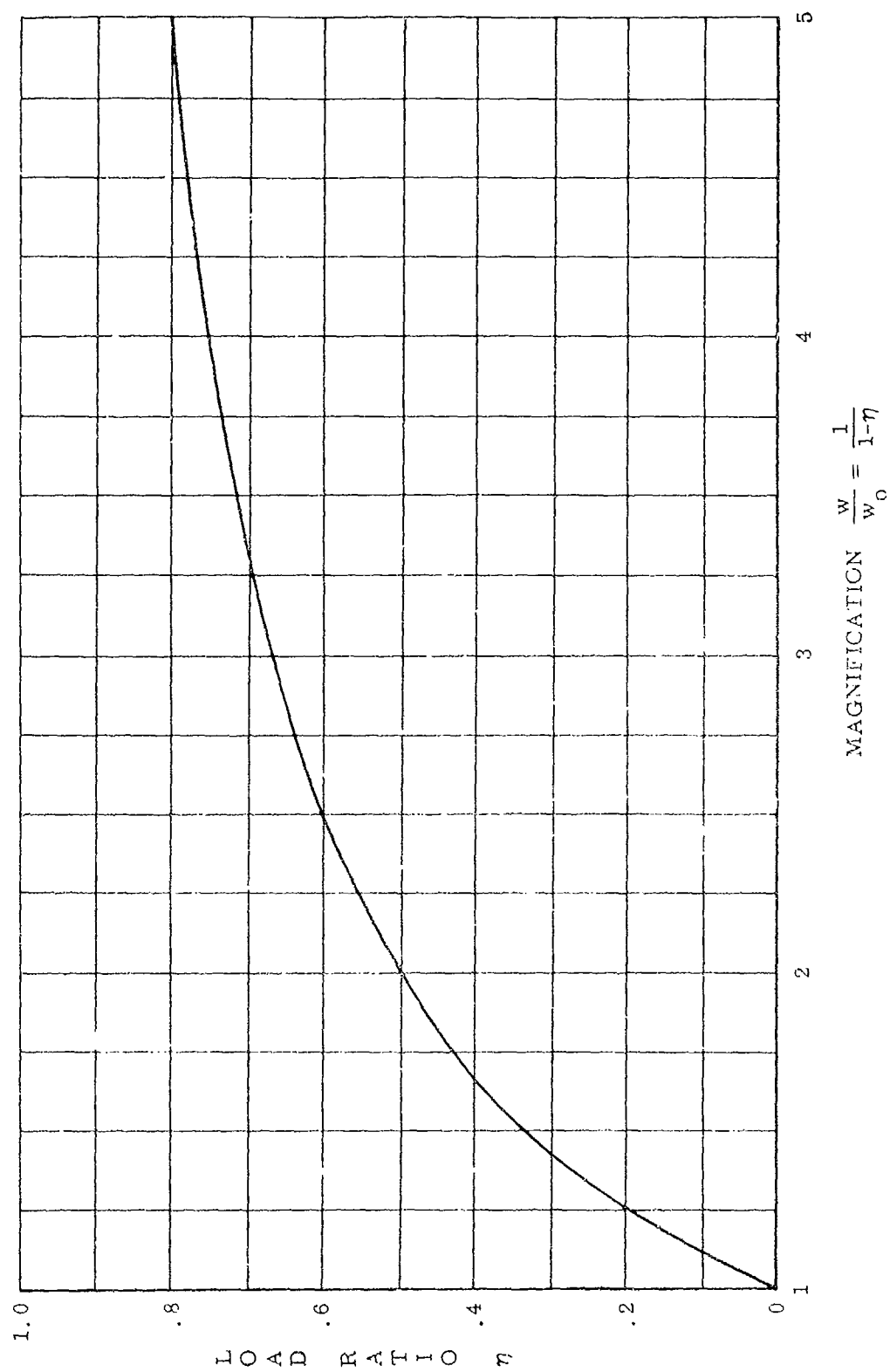


FIGURE 2.2.1-1 MAGNIFICATION OF DEFORMA ION MODES

2.2.2 (Cont'd)

$$\text{i.e.} \quad \kappa = \sum C_i \kappa_i \quad (2a)$$

$$\text{where} \quad (\phi \kappa_i)'' + \lambda_i r \kappa_i = 0 \quad (2b)$$

If the unrestrained column is not straight because of initial eccentricities, thermal gradients, and lateral loads, then the moment acting on the column is proportional to the change in curvature and the equilibrium equation becomes, for an end loaded column ($r = 1$),

$$(\phi \kappa)'' + \lambda \kappa = \left[\phi (\kappa_i + \kappa_T) \right]'' \quad (3)$$

where κ_i is the curvature caused by the lateral load and κ_T is the curvature caused by the thermal gradient.

Expressing the solution as a weighted sum of the characteristic curvatures (Eq. (2a)) which satisfy the homogeneous equation and manipulating Eq. (3) results in the solution for the curvature of an axially loaded column in terms of the initial curvature and the load ratio.

$$\kappa = \sum \frac{b_i}{1 - \eta_i} \kappa_i \quad (4)$$

where $b_i \kappa_i$ corresponds to the i^{th} component of the initial curvature of the column due to the lateral load and temperature. Thus the column deforms under axial load by having each component of the initial curvature increase by a magnification factor $1/(1 - \eta_i)$. Correspondingly the slopes and lateral deflection modes increase by this same factor.

Since the initial deformations are expressible as an infinite series of the characteristic curvatures, it becomes expeditious to employ the initial curvatures and to integrate this infinite series to obtain the expression for the slope which is employed in obtaining the axial shortening. Employing the initial slope or lateral deflection will result in a slower convergence of the series expressing the axial shortening and would reduce the accuracy of the approximate procedure employed to solve the compatibility equation.

2.2.3 Determination of Deformation Modes and Buckling Loads

The deflection modes (w) of the column are obtained by solving the homogeneous equilibrium Eq. (1) of Paragraph 2.2.1. The buckling loads (F_i) are those values of the axial load which result in non-trivial solutions of the equation. The technique of obtaining these items is illustrated for a column of constant bending stiffness ($\phi = 1$).

The homogeneous equilibrium equation for constant bending stiffness (EI) is

$$\frac{d^4 w}{dx^4} + \frac{F}{EI} \frac{d^2 w}{dx^2} = w^{IV} + \lambda w'' = 0 \quad (1)$$

The general solution is

$$w = c_1 + c_2 x + c_3 \cos \sqrt{\lambda} x + c_4 \sin \sqrt{\lambda} x \quad (2)$$

where c_1, c_2, c_3, c_4 are constants to be determined by the boundary conditions.

2.2.3 (Cont'd)

Solutions for various boundary conditions can be found in various texts. The simple-simple, clamped-clamped, and simple-clamped are summarized below. In addition the method of determining the magnitude of the initial curvature modes is indicated.

(1) Simple-Simple Column

The boundary conditions are

$$w(0) = w(l) = 0$$

$$w''(0) = w''(l) = 0$$

$$\therefore w_i = a_i \sin \frac{i \pi x}{l} = a_i \sin \sqrt{\lambda_i} x \quad (3a)$$

$$\text{where } w = \sum w_i$$

$$\frac{F_i}{EI} = \lambda_i = \left(\frac{i \pi}{l} \right)^2 \quad (4a)$$

$$x_i = \sin \frac{i \pi x}{l} \quad (5a)$$

$$\int_0^l x_i^2 dx = l/2$$

$$C_i = \frac{\int_0^l w'' x_i dx}{\int_0^l x_i^2 dx} = \frac{2}{l} \int_0^l w'' \sin \frac{i \pi x}{l} dx \quad (6a)$$

$$\text{where } w'' = C_i x_i$$

(2) Clamped-Clamped Column

The boundary conditions are

$$w(0) = w(l) = 0$$

$$w'(0) = w'(l) = 0$$

This results in two different types of modes. Modes which are symmetrical about the mid-length of the column and modes which are anti-symmetrical.

For symmetrical modes

$$w_{is} = a_i \left(1 - \cos \frac{2 i \pi x}{l} \right) \quad (3b)$$

2.2.3 (Cont'd)

$$\frac{F_i}{EI} = \lambda_i = \frac{4 i^2 \pi^2}{\ell^2} \quad (4b)$$

$$x_{is} = \cos \frac{2 i \pi x}{\ell} \quad (5b)$$

$$C_{is} = \frac{\int_0^\ell w'' x_i dx}{\int_0^\ell x_i^2 dx} = \frac{2}{\ell} \int_0^\ell w'' \cos \frac{2 i \pi x}{\ell} dx \quad (6b)$$

For anti-symmetrical modes

$$w_i = a_i \left(\frac{2\mu_i \frac{x}{\ell} - \sin 2\mu_i \frac{x}{\ell}}{2\mu_i - \sin 2\mu_i} - \frac{1 - \cos 2\mu_i \frac{x}{\ell}}{1 - \cos 2\mu_i} \right)$$

$$x_i = \frac{\sin 2\mu_i \frac{x}{\ell}}{2\mu_i - \sin 2\mu_i} - \frac{\cos 2\mu_i \frac{x}{\ell}}{1 - \cos 2\mu_i} \quad \text{where } \mu_i = \tan \mu_i$$

An alternate form, employing the mid-length of the column as an origin, results in a simpler result which is identical to the joining of two simple-clamped columns.

$$w_{ia} = a_i \left(\frac{\frac{\sin \frac{\mu_i}{2} \frac{x}{\ell}}{\sin \frac{\mu_i}{2}} - \frac{x}{2\ell}}{\sin \frac{\mu_i}{2}} \right) \quad (3c)$$

$$\frac{F_{ia}}{EI} = \lambda_i = \left(\frac{\mu_i}{\ell/2} \right)^2 = \frac{4\mu_i^2}{\ell^2} \quad (4c)$$

$$\text{where } \tan \mu_i = \mu_i$$

$$\text{and } \mu_1 = 1.43 \quad \mu_n = \left(\frac{2n+1}{2} \right) \pi$$

$$x_{ia} = \sin \frac{\mu_i}{2} \frac{x}{\ell} \quad (5c)$$

$$C_{ia} = \frac{\int_0^{\ell/2} w'' x_i dx}{\int_0^{\ell/2} x_i^2 dx} = \frac{\ell}{\ell \left(\frac{1}{4} - \frac{\cos \mu_i}{\mu_i} \right)} \int_0^{\ell/2} w'' \sin \frac{\mu_i}{2} \frac{x}{\ell} dx \quad (6c)$$

2.2.3 (Cont'd)

(3) Simple Clamped Column

The boundary conditions are

$$w(0) = w''(0) = 0$$

$$w(l) = w'(l) = 0$$

$$w_i = a_i \left(\frac{\sin \mu_i \frac{x}{l}}{\sin \mu_i} - \frac{x}{l} \right) \quad (3d)$$

$$\text{where } \tan \mu_i = \mu_i$$

$$\frac{F_i}{EI} = \lambda_i = \frac{\mu_i^2}{l^2} \quad (4d)$$

$$x_i = \sin \mu_i \frac{x}{l} \quad (5d)$$

$$C_{ia} = \frac{\int_0^l w'' x_i dx}{\int_0^l x_i^2 dx} = \frac{1}{l \left(\frac{1}{2} - \frac{\cos 2\mu_i}{\mu_i} \right)} \int_0^l w'' \sin \mu_i \frac{x}{l} dx \quad (6d)$$

The deformation modes and buckling loads are required in the determination of the end load and deflection of a restrained column as indicated in Subsection 2.3. If the boundary conditions or stiffness distribution do not conform to the case summarized above, then the buckling loads and modes must be determined for the column considered. Special cases may be found in various text and recourse must be taken to approximate solutions when the column becomes more involved.

The method of solution for a column with a linear material, proposed in this report, is quite general and can handle the more complex configurations provided the initial deflection can be described by characteristic eigenvectors with known eigenvalues.

2.2.4 Compatibility Equation

The axially unrestrained column (Figure 2.1-1b) will deform when subjected to lateral loads and temperature. This will cause an axial motion of the ends of the column of Δ_{00} .

$$\Delta_{00} = l\alpha T - \Delta_0 \quad (1a)$$

where $\alpha T = \epsilon_T$ is the unrestrained axial strain due to temperature. If the strains are not uniform then

$$\alpha T = \frac{1}{l} \int_0^l \frac{\int \alpha T dA}{\int dA} dx \quad (1b)$$

2.2.4 (Cont'd)

Δ_o = axial shortening of the column due to the lateral load and temperature gradients through the cross sections

$$\Delta_o = \frac{1}{2} \int_0^l (w_T' + w_q')^2 dx \quad (1c)$$

The application of the axial load causes the column to shorten due to the axial strain $\frac{F\ell}{AE}$ and additional axial shortening $(\Delta - \Delta_o)$. The final movement of the ends must be equal to the deformation of the axial spring F/k .

$$\therefore \alpha T - \Delta_o - \frac{F\ell}{AE} - (\Delta - \Delta_o) = \frac{F}{k} \quad (2a)$$

Dividing by the quantity $\ell\epsilon_1$ results in

$$\frac{\alpha T - \frac{\Delta_o}{\ell}}{\epsilon_1} - \frac{F}{AE\epsilon_1} - \frac{\frac{\Delta}{\ell} - \frac{\Delta_o}{\ell}}{\epsilon_1} - \frac{F}{k\ell\epsilon_1} = 0 \quad (2b)$$

$$\text{where } \epsilon_1 = \frac{F\ell}{AE} = \frac{\lambda_1 EI}{AE} = \lambda_1 \rho^2 \quad (2c)$$

Noting that

$$\frac{F}{AE\epsilon_1} = \frac{\sigma A}{AE\epsilon_1} = \frac{E\epsilon A}{AE\epsilon_1} = \frac{\epsilon}{\epsilon_1} = \eta_1 \quad (3a)$$

and

$$\frac{F}{k\ell\epsilon_1} = \frac{E\epsilon A}{k\ell\epsilon_1} = \frac{EA}{k\ell} = \eta_1 \quad (3b)$$

and letting

$$\frac{\alpha T - \Delta_o/\ell}{\epsilon_1} = b \quad (4)$$

and evaluating the axial shortening

$$\Delta_o = \frac{1}{2} \int_0^l (w_T' + w_q')^2 dx = \frac{1}{2} \int_0^l (w_o')^2 dx = \frac{1}{2} \int_0^l (\Sigma w_{io}')^2 dx \quad (5a)$$

but because of the orthogonality of the characteristic slopes (Eq. (3) of Paragraph 2.2.1) we obtain

$$\Delta_o = \frac{1}{2} \int_0^l \Sigma (w_{io}')^2 dx = \frac{1}{2} \Sigma \int_0^l (w_{io}')^2 dx \quad (5b)$$

2.2.4 (Cont'd)

and

$$\begin{aligned}\Delta - \Delta_0 &= \frac{1}{2} \int_0^L \left[(w')^2 - (w'_0)^2 \right] dx = \frac{1}{2} \int_0^L \Sigma (w'_{i0})^2 \left[\left(\frac{1}{1 - \eta_1} \right)^2 - 1 \right] dx \\ &= \frac{1}{2} \Sigma \left[\left(\frac{1}{1 - \eta_1} \right)^2 - 1 \right] \int_0^L (w'_{i0})^2 dx\end{aligned}\quad (5c)$$

letting

$$\frac{\Delta_i}{L \epsilon_1} = \frac{\frac{1}{2} \int_0^L (w'_{i0})^2 dx}{L \epsilon_1} = d_i \quad (6a)$$

and

$$\left(\frac{1}{1 - \eta_1} \right)^2 - 1 = c_i \quad (6b)$$

we obtain

$$\frac{\Delta_0}{L \epsilon_1} = \Sigma d_i \quad (6c)$$

and

$$\frac{\Delta - \Delta_0}{L \epsilon_1} = \Sigma d_i c_i \quad (6d)$$

Substituting Eqs. (3a), (3b), (4), and (6d) into Eq. (2b) results in

$$b - \eta_1 - \Sigma d_i c_i - \frac{EA}{kL} \eta_1 = 0 \quad (7)$$

The above equation is quite difficult to solve, but the solution can be simplified if it is noted that the expression $d_i c_i$ denotes the increase in axial deformation due to the growth of the i^{th} mode of deformation. The first mode is magnified to a much greater degree than the higher modes when the axial load is compression. In fact the first mode becomes predominant as the axial load approaches the first buckling load. The approximate method utilizes this fact in approximating the value of $\Sigma d_i c_i$ by expanding $d_i c_i$ (for $i = 1$) as a power series and only employing the most significant term and by approximating the infinite series by a finite series. This approximation method is not recommended for tension loads, since the approximation for c_i and the infinite series may be incorrect.

2.2.4 (Cont'd)

Noting that

$$\varphi_i = \left(\frac{1}{1 - \eta_i} \right)^2 - 1 = 1 + 2\eta_i + 3\eta_i^2 + \dots - 1 \sim 2\eta_i \quad (8a)$$

and that $1 > \eta_1 > \eta_i \quad (i \geq 2)$

$$\therefore \sum_{i=1}^{\infty} d_i \varphi_i = d_1 \varphi_1 + \sum_{i=2}^{\infty} d_i \varphi_i \text{ is approximately } d_1 \varphi_1 + 2 \sum_{i=2}^n \eta_i d_i \quad (8b)$$

(8c)

where n is a sufficiently large integer.

Equation (7) can thus be approximated by

$$b - d_1 \varphi_1 - \left(1 + \frac{EA}{k} + 2 \sum_{i=2}^n \frac{\eta_i}{\eta_1} d_i \right) \eta_1 = 0 \quad (9a)$$

Letting

$$\Phi = 1 + \frac{EA}{k} + 2 \sum_{i=2}^n \frac{\eta_i}{\eta_1} d_i \quad (9b)$$

we obtain

$$\frac{b}{\Phi} - \frac{d_1}{\Phi} \varphi_1 - \eta_1 = 0 \quad (9c)$$

or

$$\bar{b} - \bar{d}_1 \varphi_1 - \eta_1 = 0 \quad (9d)$$

where

$$\bar{b} = \frac{b}{\Phi} \quad (9e)$$

and

$$\bar{d}_1 = \frac{d_1}{\Phi} \quad (9f)$$

The value of the axial strain parameter (η_1) can be determined from a graphical plot (Figure 2.2.4-1 and -2) of the variation of \bar{b} with η_1 for various \bar{d}_1 . This value of η_1 can then

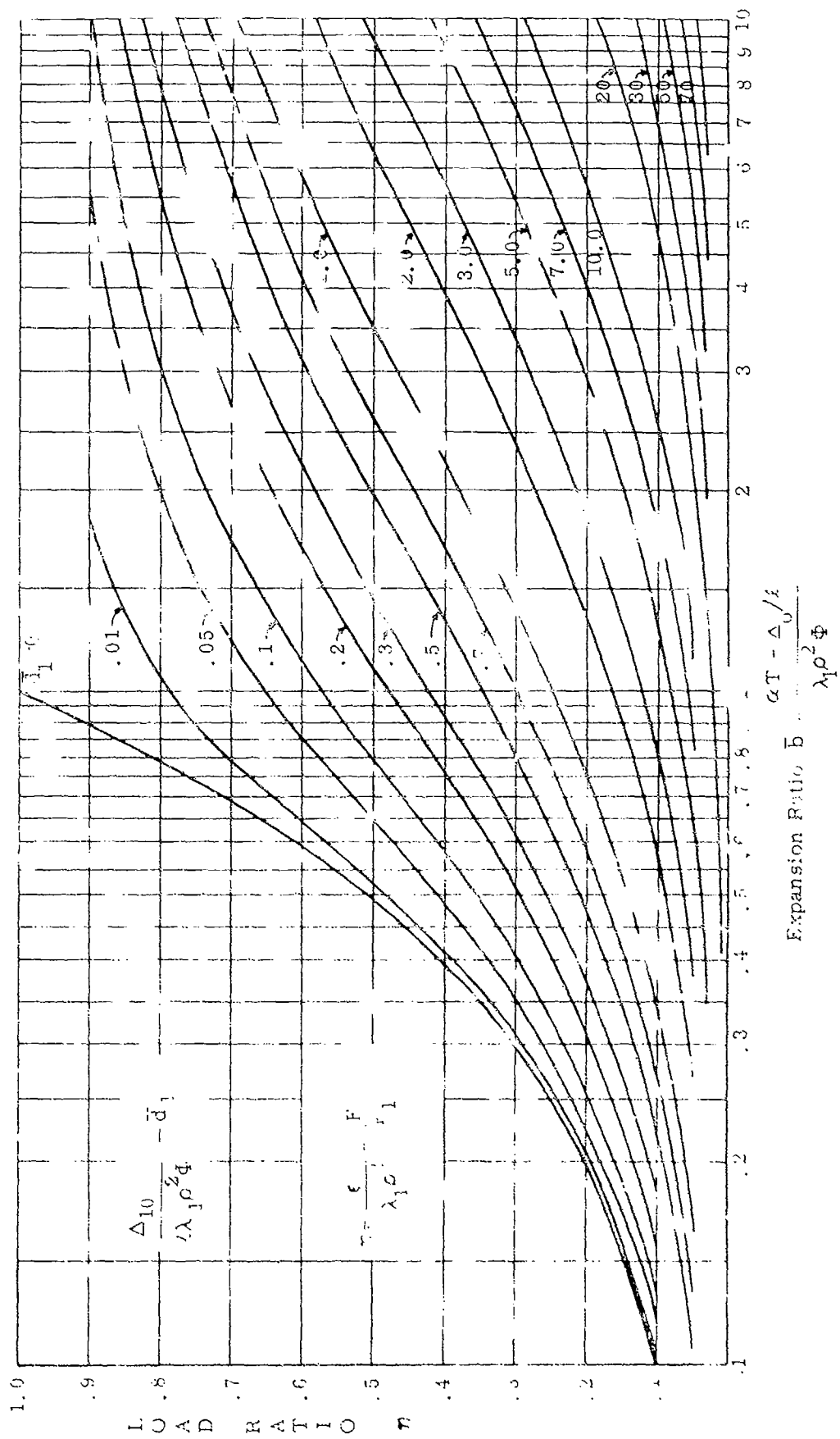


FIGURE 2.2.4-1 AXIAL LOAD IN A RESTRAINED COLUMN

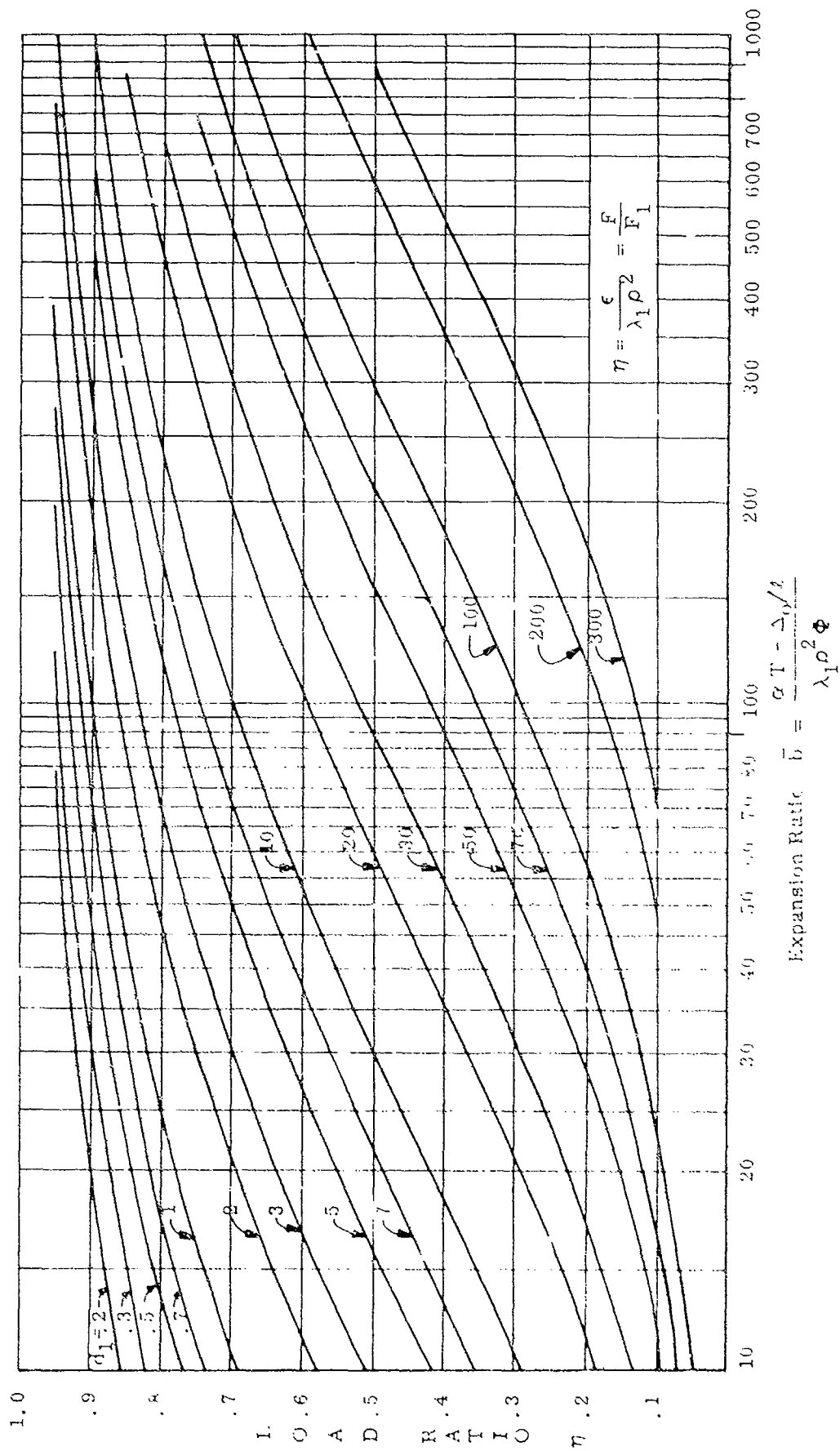


FIGURE 2.2.4-2 AXIAL LOAD IN A RESTRAINED COLUMN

2.2.4 (Cont'd)

be employed to obtain the axial load acting on the column and the deformations of the column. It should be noted that the approximate method results in an upper bound on the axial load since it ignores a small part of the change of axial motion due to the higher modes.

2.3 PROCEDURE

The information and technique needed to solve the problem are enumerated in the following sections.

2.3.1 Determination of Nondimensional Parameters

By the definition Eq. (6a) of Paragraph 2.2.4:

$$d_1 = \frac{\frac{1}{2} \int_0^l (w'_{10})^2 dx}{l \epsilon_1} = \frac{\Delta_1}{l \lambda_1 \rho^2}$$

Equation (9.1.2 - 3c) of Reference 2-2 indicates that

$$\lambda_1 = \frac{F_1}{EI} = \frac{\int_0^l \phi (w''_1)^2 dx}{\int_0^l (w'_1)^2 dx} = \frac{\int_0^l \phi C_1^2 x_1^2 dx}{2 \Delta_1} \quad (1a)$$

$$\therefore \Delta_1 = \frac{C_1^2 \int_0^l \phi x_1^2 dx}{2 \lambda_1} \quad (1b)$$

and

$$d_1 = \frac{C_1^2 \int_0^l \phi x_1^2 dx}{2 l \lambda_1 \lambda_1 \rho^2} \quad (2a)$$

Noting that

$$\frac{\eta_1}{\eta_1} = \frac{\epsilon/\epsilon_1}{\epsilon/\epsilon_1} = \frac{\epsilon_1}{\epsilon_1} = \frac{\lambda_1 \rho^2}{\lambda_1 \rho^2} = \frac{\lambda_1}{\lambda_1} \quad (3)$$

$$\therefore \frac{\eta_1}{\eta_1} d_1 = \frac{C_1^2 \int_0^l \phi x_1^2 dx}{l \lambda_1^2 \rho^2} \quad (4a)$$

2.3.1 (Cont'd)

d_i and $\frac{\eta_i d_i}{\eta_1}$ will be evaluated for the chosen boundary conditions and for constant EI ($\varphi=1$) utilizing Eqs. (5) and (6) of Paragraph 2.2.3.

(1) Simple-Simple Column

$$d_i = \frac{C_i^2}{2l \left(\frac{\pi}{l}\right)^2 \left(\frac{i\pi}{l}\right)^2} \frac{l/2}{\rho^2} = \frac{l^4}{4\pi^4 \rho^2} \left(\frac{C_i^2}{i^2}\right) \quad (2b)$$

$$2 \frac{d_i \eta_i}{\eta_1} = \frac{C_i^2}{l \left(\frac{i\pi}{l}\right)^4} \frac{l/2}{\rho^2} = \frac{l^4}{2\pi^4 \rho^2} \left(\frac{C_i^2}{i^4}\right) \quad (4b)$$

(2) Clamped-Clamped Column

Symmetrical

$$d_{1s} = \frac{C_{1s}^2}{2l \rho^2 \left(\frac{2\pi}{l}\right)^2 \left(\frac{2i\pi}{l}\right)^2} \frac{l/2}{\rho^2} = \frac{l^4}{64\pi^4 \rho^2} \left(\frac{C_{1s}^2}{i^2}\right) \quad (2c)$$

$$2 \frac{d_{1s} \eta_{1s}}{\eta_1} = \frac{C_{1s}^2}{l \rho^2 \left(\frac{2i\pi}{l}\right)^4} \frac{l/2}{\rho^2} = \frac{l^4}{32\pi^4 \rho^2} \left(\frac{C_{1s}^2}{i^4}\right) \quad (4c)$$

Antisymmetrical

$$d_{ia} = \frac{C_{ia}^2 \left(\frac{1}{2} - \frac{2 \cos \mu_i}{\mu_i}\right)}{2l \rho^2 \left(\frac{2\pi}{l}\right)^2 \left(\frac{2\mu_i}{l}\right)^2} = \frac{l^4 C_{ia}^2 \left(\frac{1}{2} - \frac{2 \cos \mu_i}{\mu_i}\right)}{32 \pi^2 \rho^2 \mu_i^2} \quad (2d)$$

$$2 \frac{d_{ia} \eta_{ia}}{\eta_1} = \frac{C_{ia}^2 \left(\frac{1}{2} - \frac{2 \cos \mu_i}{\mu_i}\right)}{l \rho^2 \left(\frac{2\mu_i}{l}\right)^4} = \frac{l^4 C_{ia}^2 \left(\frac{1}{2} - \frac{2 \cos \mu_i}{\mu_i}\right)}{16 \rho^2 \mu_i^4} \quad (4d)$$

2.3.1 (Cont'd)

(3) Clamped-Simple Column

$$d_1 = \frac{C_1^2 \left(\frac{1}{2} - \frac{\cos \mu_1}{\mu_1} \right)}{2 \rho^2 \left(\frac{1.43\pi}{l} \right)^2 \left(\frac{\mu_1}{l} \right)^2} = \frac{l^4 C_1^2 \left(\frac{1}{2} - \frac{\cos 2\mu_1}{\mu_1} \right)}{4.1 \pi^2 \rho^2 \mu_1^2} \quad (2e)$$

$$2 \frac{d_1 \eta_1}{\eta_1} = \frac{C_1^2 \left(\frac{1}{2} - \frac{\cos 2\mu_1}{\mu_1} \right)}{l \rho^2 \left(\frac{\mu_1}{l} \right)^2} = \frac{l^4}{\rho^2} \frac{C_1 \left(\frac{1}{2} - \frac{\cos \mu_1}{\mu_1} \right)}{\mu_1^4} \quad (4e)$$

(4) Summary

Equation (9a) of Paragraph 2.2.4 can then be summarized for columns of constant EI as follows:

Simple-Simple

$$\frac{\alpha T - \Delta_o / l}{\frac{\pi^2}{l^2} \rho^2} - \left(\frac{l^4 C_1^2}{4\pi^4 \rho^2} \right) \phi_1 - \left(1 + \frac{EA}{kl} + \frac{l^4}{2\pi^4 \rho^2} \sum_{i=2}^n \frac{C_i^2}{i^2} \right) \eta_1 = 0 \quad (5a)$$

Clamped-Clamped Column

$$\frac{\alpha T - \Delta_o / l}{\frac{4\pi^2}{l^2} \rho^2} - \left(\frac{l^4 C_1^2}{64\pi^4 \rho^2} \right) \phi_1 - \left[1 + \frac{EA}{kl} + \frac{l^4}{32\pi^4 \rho^2} \sum_{i=1}^n \frac{C_{18}^2}{i^4} + \frac{l^4}{16\rho^2} \sum_{i=1}^n \frac{C_i^2 \left(\frac{1}{2} - \frac{\cos 2\mu_i}{\mu_i} \right)}{\mu_i^4} \right] \eta_1 = 0 \quad (5b)$$

Simple-Clamped Column

$$\frac{\alpha T - \Delta_o / l}{\frac{2.05\pi^2}{l^2} \rho^2} - \left(\frac{0.07081 l^4 C_1^2}{\pi^4 \rho^2} \right) \phi_1 - \left[1 + \frac{EA}{kl} + \frac{l^4}{\rho^2} \sum_{i=2}^n \frac{C_i^2 \left(\frac{1}{2} - \frac{\cos 2\mu_i}{\mu_i} \right)}{\mu_i^4} \right] \eta_1 = 0 \quad (5c)$$

2.3.2 Computational Procedure

The following computational procedure is recommended to obtain the approximate solution for the axial load and lateral deflection of a restrained column.

- (1) Determine the axial expansion of the column due to temperature and the lateral load, assuming, at this step, that the restraint against actual expansion is removed.

$$\ell \alpha T - \Delta_0 = \int_0^\ell \epsilon_T dx - \int_0^\ell (w_0')^2 dx \quad (1)$$

where

$$w_0 = w_q + w_T$$

w_q = lateral deflection due to lateral loads (usually obtained from reference texts)

w_T = lateral deflection due to thermal gradients (method of determining w_T indicated in Section 4 of Reference 2-2).

ϵ_T = axial expansion due to temperature (method of determining ϵ_T indicated in Section 4 of Reference 2-2).

- (2) Calculate the critical axial strain from Eq. (2c) of Paragraph 2.2.4)

$$\epsilon_1 = \lambda_1 \frac{I}{A} = \lambda_1 \rho^2$$

$$\text{(e.g. } \epsilon_1 = \frac{\pi^2}{2} \rho^2 \text{ for simple-simple, } \phi = 1)$$

- (3) Expand the initial curvature ($w_0'' = \sum C_i x_i$), due to the lateral load and temperature, in terms of the characteristic curvatures of the column. A sufficient number of terms of the series should be taken to ensure accuracy. The coefficients of the characteristic curvatures are obtained by the general equation

$$C_i = \frac{\int_0^\ell w_0'' x_i dx}{\int_0^\ell c x_i^2 dx} \quad (2)$$

Formulas for constant EI ($c = 1$) are presented in Eqs. (6a), (6b), (6c), and (6d) of Paragraph 2.2.3. Tables to evaluate the integrals of Eqs. (6a) and (6b) of Paragraph 2.2.3 are presented in Tables 2.3.3.1, 2-1 and -2.

- (4) Evaluate the pertinent parameters

$$(a) \quad d_1 = \frac{\Delta_{10}}{\ell \epsilon_1} = \frac{C_1^2 \int c x_1^2 dx}{2 \rho^2 \lambda_1 \lambda_1'} \quad \text{(Reference Eq. (2a) of Paragraph 2.3.1)}$$

2.3.2 (Cont'd)

Appropriate formulas for d_i are presented for columns of constant EI and for the chosen boundary conditions in Paragraph 2.3.1.

$$(e.g. \quad d_i = \frac{l^4}{4\pi^4 \rho^2} \frac{C_i^2}{i^2} \quad \text{for simple-simple column}).$$

$$(b) \quad \Phi = 1 + \frac{EA}{kl} + 2 \sum_{i=2}^n \frac{\eta_i}{\eta_1} d_i \quad (\text{Reference Eq. (9b) of Paragraph 2.2.4})$$

Values of Φ are indicated in Eq. (5) of Paragraph 2.3.1 for constant EI.

$$(e.g. \quad \Phi = 1 + \frac{EA}{kl} + \frac{l^4}{2\pi^4 \rho^2} \sum_{i=2}^n \frac{C_i^2}{i^4} \quad \text{for simple-simple column})$$

$$(c) \quad \bar{d}_1 = \frac{d_1}{\Phi} \quad (\text{Reference Eq. (9f) of Paragraph 2.2.4})$$

$$(d) \quad \bar{b} = \frac{\alpha T - \Delta_{o/l}}{\epsilon_1 \Phi} = \frac{\alpha T - \Delta_{o/l}}{\lambda_1 \rho^2 \Phi} \quad (\text{Reference Eq. (9e) of Paragraph 2.2.4})$$

(5) Enter Figure 2.2.4-1 or -2 with the appropriate values of \bar{b} and \bar{d}_1 and determine η_1 .

(6) The axial load is then determined from Eq. (3a) of Paragraph 2.2.4.

$$F = \sigma A = E \frac{\epsilon}{\epsilon_1} \epsilon_1 A = EA \lambda_1 \rho^2 \eta_1 \quad (3)$$

(7) The lateral deflection of the column could likewise be determined as

$$w = \sum \frac{w_{oi}}{1 - \eta_i} \quad (4a)$$

$$\text{where the value of } \eta_i = r_1 \frac{\lambda_1}{\lambda_i} \quad (4b)$$

and the value of $\frac{1}{1-\eta_i}$ can be calculated or determined with the aid of Figure 2.2.1-1.

$$(e.g. \quad w = \frac{2}{\pi^2} \sum \left(\frac{l^2 C_i}{2} \right) \frac{\sin (l \pi x / l)}{1^2 - \eta_1} \quad \text{for pinned end column}).$$

2.3.2 (Cont'd)

(8) A slightly better approximation to the value of η_1 can be obtained by employing

$$\bar{b} = \frac{\frac{\alpha T - \Delta_o/l}{\lambda_1 \rho^2} - \left(\frac{\Delta_o}{\lambda_1 \rho^2} - \sum_{i=1}^n d_i \right)}{\Phi} \quad (5)$$

which corrects for the error in approximating the initial shortening by a finite series. The additional shortening of the higher modes is still approximated by the first term of the series in order to simplify the solution of the compatibility equation.

(9) The graphs can also be employed to determine the approximate average temperature rise (with no cross-sectional gradient) required to produce a maximum permissible lateral deflection. The permissible ratio of maximum to initial lateral deflection $\left(\frac{w_{\max}}{w_o} \right)$ can be employed with Figure 2.2.1-1 to estimate a value of η_1 which can be utilized with \bar{d}_1 in Figures 2.2.4 to determine a value of \bar{b} which could then be employed to calculate αT . (If the higher modes of the initial deflection are significant, then they must be included as indicated in Eq. (4a)). The initial eccentricity will permit the calculation of Φ and \bar{d}_1 . Assuming a value of η_1 or of αT will permit the determination of the other from Figure 2.2.4. The value of η_1 could then be used to determine w which could be plotted against αT ; αT could also be plotted against the axial load by employing Eq. (3a) of Paragraph 2.2.4. A typical plot is shown in Figure 2.3.2-1.

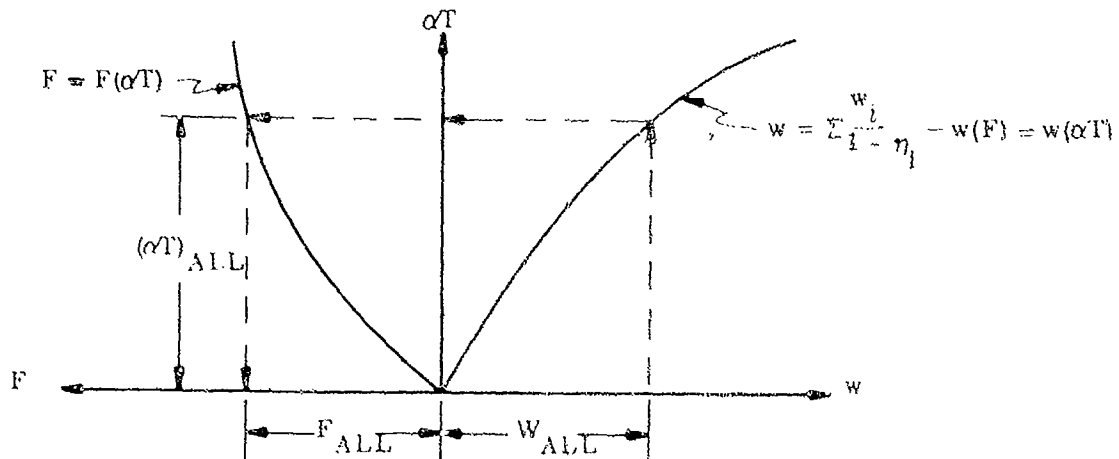


FIGURE 2.3.2-1 AXIAL LOAD AND LATERAL DEFLECTION VS AVERAGE TEMPERATURE RISE

2.3.3 Initial Deformations

The initial deformations before the end load is applied must be determined to a fair degree of accuracy. Two methods are presented herein. The first procedure assumes that an analytical expression for the initial lateral deflections is available as a polynomial, while the second method assumes that the lateral deflection is known only at a finite number of points (e.g., determined experimentally or by computations).

2.3.3.1. Polynomial Representation

The lateral deformation of a beam can be determined by integrating the equilibrium equation (Eq. (3) of Paragraph 2.2.2) with $\lambda = 0$ and employing the given boundary conditions. Solutions for lateral loading are available in literature for the type of boundary conditions con-

sidered (e.g., $w_q = \frac{q l^4}{24 EI} \left[\left(\frac{x}{l} \right) - 2 \left(\frac{x}{l} \right)^3 + \left(\frac{x}{l} \right)^4 \right]$ for a simple-simple beam with uni-

form lateral load). The slopes and curvatures can be obtained by differentiating these polynomial expressions. The lateral deflection due to thermal gradients is not as readily available but can be derived quite simply by utilizing the equations of Section 4 of Reference 2-2. The technique is illustrated for the simple-simple and clamped-clamped beams.

2.3.3.1.1 Lateral Deformation of Beams by Temperature Gradients

Let the thermal gradient through the depth of the beam be expressed as a polynomial in the spanwise dimension,

$$\text{e.g.} \quad \frac{\alpha T_0 - \alpha T_1}{h} = - \frac{\Delta \alpha T}{h} = \frac{1}{l^2} \sum m_j \left(\frac{x}{l} \right)^j = \frac{1}{l^2} \sum m_j \xi^j \quad (1)$$

where T_1 and T_0 are the temperatures on the positive and negative side of the column with a linear gradient (see Figure 2.1-1).

If the structure is statically determinate, as in a simple-simple beam, then the thermal gradient represents the total curvature (w''_T) since no redundant forces are produced by the temperature. The slope and deflection can then be obtained by integrating this curvature and employing the given boundary conditions.

$$\therefore w = \frac{1}{2} \int_0^x \int_0^{x_0} \sum m_j \left(\frac{x}{l} \right)^j dx dx = \int_0^{\xi} \int_0^{\xi_0} \sum m_j \xi^j d\xi d\xi \quad (2)$$

$$w = \sum \frac{m_j \xi^{j+2}}{(j+1)(j+2)} + c_1 \xi + c_2$$

For simple supports $w(0) = w(l) = 0$

$$\therefore w = \sum \frac{m_j}{(j+1)(j+2)} (\xi^{j+2} - \xi) \quad (3a)$$

$$\& w' = \frac{1}{l} \sum \frac{m_j}{(j+1)(j+2)} [(j+2) \xi^{j+1} - 1] \quad (3b)$$

2.3.3.1.1 (Cont'd)

If the structure is statically indeterminate, as in the clamped-clamped beam, then the curvature is affected by the curvature caused by the redundant loads. Equations (4.2.2.5 - 7 and -8) of Reference 2-2 can be employed to determine the deformation of the clamped beams where subjected to a thermal gradient. The thermal gradient $\frac{\Delta \alpha T}{h} =$

$-m_j \xi^j / i^2$ induces redundant loads P_0 and M_0 and the following total deformations result:

$$w_T'' = \frac{1}{2} \sum m_j \left[\xi^j - \frac{6j\xi}{(j+1)(j+2)} - \frac{2(1-j)}{(j+1)(j+2)} \right] \quad (4a)$$

$$w_T' = \frac{1}{i} \sum m_j \left[\frac{\xi^{j+1}}{j+1} - \frac{3j\xi^2}{(j+1)(j+2)} - \frac{2(1-j)\xi}{(j+1)(j+2)} \right] \quad (4b)$$

$$w_T = \sum \frac{m_j}{(j+1)(j+2)} \left[\xi^{j+2} - j\xi^3 - (1-j)\xi^2 \right] \quad (4c)$$

2.3.3.1.2. Expansion in Characteristic Curvatures

The polynomial definition of curvature can be converted to a eigenvector expansion by means of Eqs. (2a) of Paragraph 2.2.2 and (2) of Paragraph 2.3.2.

$$w_T'' + w_q'' = w_0'' = \sum C_i x_i \quad (\text{Refer to Eq. (2a) of Paragraph 2.2.2})$$

$$C_i = \frac{\int_0^l \phi w_0'' x_i dx}{\int_0^l \phi x_i^2 dx} \quad (\text{Refer to Eq. (2) of Paragraph 2.3.2})$$

For the simple-simple column with constant EI, this results in the simple Fourier sine expansion of the initial curvature.

$$C_i = \frac{2}{l} \int_0^l w_0'' \sin \frac{i\pi x}{l} dx = \frac{2}{l^2} \sum m_j \int_0^1 \xi^j \sin i\pi \xi d\xi \quad (1a)$$

where $w_0'' = \frac{1}{2} \sum m_j \xi^j$

The clamped-clamped column with constant EI and symmetrical loading results in a Fourier cosine expansion.

$$C_{is} = \frac{2}{l} \int_0^l w_{0s}'' \cos \frac{2i\pi x}{l} dx = \frac{2}{l^2} \sum m_j \int_0^1 \xi^j \cos 2i\pi \xi d\xi \quad (1b)$$

The curvature coefficients for the clamped-simple and the clamped-clamped columns with anti-symmetrical load must be similarly evaluated.

2.3.3.1.2 (Cont'd)

Tables 2.3.3.1.2-1 and -2 present the values of the integrals for the simple-simple and the symmetrical clamped-clamped columns.

TABLE 2.3.3.1.2-1 FOURIER EXPANSION OF MONOMIAL FOR SIMPLE-SIMPLE COLUMN

		S(j, i)					
j \ i		1	2	3	4	5	6
0		.6366	0	.2122	0	.12732	0
1		.3183	-.1591	.1061	-.07958	.06366	-.05305
2		.1893	-.1591	.1013	-.07958	.06262	-.05303
3		.1248	-.1350	.09894	-.07655	.06211	-.05215
4		.08814	-.1108	.09241	-.07353	.06061	-.05125
5		.06541	-.09078	.08383	-.06988	.05862	-.05011
6		.05038	-.07497	.07489	-.06560	.05629	-.04872
7		.03995	-.06258	.06647	-.06099	.05368	-.04712
8		.03243	-.05280	.05889	-.05631	.05088	-.04537
9		.02683	-.04503	.05223	-.05176	.04799	-.04350
10		.02256	-.03877	.04644	-.04748	.04510	-.04155

$$w'' = \sum C_i \sin \frac{i\pi x}{l} = \frac{1}{l^2} \sum m_j \left(\frac{x}{l} \right)^j = \frac{1}{l^2} \sum m_j \xi^j$$

$$C_i = \frac{2}{l^2} \sum m_j \int_0^1 \xi^j \sin i\pi \xi d\xi = \frac{2}{l^2} \sum_j \sum_i m_j S(j, i) \quad (2a)$$

$$S(j, i) = \int_0^1 \xi^j \sin i\pi \xi d\xi$$

$$S(j, i) = \frac{[1 + (-1)^j] (-1)^{j-2} j!}{2 (i\pi)^{j+1}} (i)^j \sum_{n=0,2,4}^{j+\frac{1}{2} \left[\frac{(-1)^j - 1}{2} \right]} \frac{(-1)^{\frac{n}{2}+1} j!}{(i\pi)^{n+1} (j-n)!} \quad (2b)$$

2.3.3.1.2 (Cont'd)

TABLE 2.3.3.1.2-2 FOURIER EXPANSION OF MONOMIAL FOR CLAMPED-CLAMPED COLUMN

		C(j, i)					
j \ i		1	2	3	4	5	6
1		0	0	0	0	0	0
2		.05066	.01267	.00563	.00317	.00203	.00141
3		.07599	.01900	.00844	.00475	.00304	.00211
4		.08592	.02437	.01107	.00627	.00403	.00280
5		.08815	.02926	.01360	.00777	.00500	.00349
6		.08669	.03337	.01595	.00920	.00596	.00416
7		.08353	.03655	.01809	.01057	.00688	.00482
8		.07967	.03883	.02000	.01185	.00777	.00546
9		.07563	.04033	.02166	.01304	.00862	.00609
10		.07167	.04120	.02308	.01414	.00942	.00669

$$w'' = \sum C_i \cos \frac{2\pi i x}{l} = \frac{1}{l^2} \sum m_j \left(\frac{x}{l}\right)^j = \frac{1}{l^2} \sum m_j \xi^j$$

$$C_i = \frac{2}{l^2} \sum m_j \int_0^1 \xi^j \cos 2i\pi \xi d\xi = \frac{2}{l^2} \sum_j \sum_i m_j (C(j, i)) \quad (3a)$$

$$C(j, i) = \int_0^1 \xi^j \cos 2i\pi \xi d\xi$$

$$C(j, i) = \frac{[1 + (-1)^j] (-1)^{j/2} j!}{2 (2i\pi)^{j+1}} + \sum_{n=1,3,5}^{j-\frac{1}{2} [(-1)^j + 1]} \frac{j!}{(2i\pi)^{j+n} (j-n)!} \quad (3b)$$

2.3.3.2 Discrete Lateral Deformation

In some instances the lateral deflection is not known as a continuous function, but is determined analytically (see Paragraph 4.2.1 of Reference 2-2) or experimentally at a discrete number of points. The deformation of "n" discrete points can be employed to obtain an approximating expansion of "n" characteristic deflection shapes. The amplitude of the "n" characteristic deflections are determined by matching the displacements at the known points. The technique is illustrated for a simple-simple column with known deflections at the 1/6 points (n = 5).

$$w(\xi_j) = w\left(\frac{x_j}{l}\right) = w_j = \sum_{i=1}^5 a_i \sin \frac{i\pi x}{l} \quad (1a)$$

The symmetry of the odd characteristic deflections and the anti-symmetry of the even characteristics can be employed to reduce the order of the simultaneous equations necessary to solve for the amplitudes (a_i).

$$\begin{aligned} \therefore \quad \frac{w_1 + w_5}{2} &= a_1 \sin \frac{\pi}{6} + a_3 \sin \frac{3\pi}{6} + a_5 \sin \frac{5\pi}{6} \\ \frac{w_2 + w_4}{2} &= a_1 \sin \frac{2\pi}{6} + a_3 \sin \frac{6\pi}{6} + a_5 \sin \frac{10\pi}{6} \\ w_3 &= a_1 \sin \frac{3\pi}{6} + a_3 \sin \frac{9\pi}{6} + a_5 \sin \frac{15\pi}{6} \\ \frac{w_1 - w_5}{2} &= a_2 \sin \frac{2\pi}{6} + a_4 \sin \frac{4\pi}{6} \\ \frac{w_2 - w_4}{2} &= a_2 \sin \frac{4\pi}{6} + a_4 \sin \frac{8\pi}{6} \end{aligned} \quad (1)$$

The solution is

$$\begin{pmatrix} a_1 \\ a_3 \\ a_5 \end{pmatrix} = \begin{pmatrix} .333 & .577 & .333 \\ .667 & 0 & .333 \\ .333 & -.577 & .333 \end{pmatrix} \begin{pmatrix} \frac{w_1 + w_5}{2} \\ \frac{w_2 + w_4}{2} \\ w_3 \end{pmatrix} \quad (1c)$$

$$a_2 = \frac{1}{3.464} (w_1 + w_2 - w_4 - w_5)$$

$$a_4 = \frac{1}{3.464} (w_1 - w_2 + w_4 - w_5)$$

2.3.3.2 (Cont'd)

It should be noted that $C_1 = -\frac{i^2 \pi^2}{l^2} a_1$ (2)

Similar solutions can be obtained for different boundary conditions and for different types of known deformations.

A rapid solution can be obtained if one assumes that the lateral deflection is primarily in the fundamental mode (e.g., uniform thermal gradient on a simple-simple column). Thus the values of the nondimensional parameters can be simply determined with a minimum amount of computation. As an example, the simple-simple column equation can be expressed as

$$\frac{\alpha T - \Delta_o/l}{\frac{\pi^2 \rho^2}{l^2}} - \left(\frac{\Delta_o/l}{\frac{\pi^2 \rho^2}{l^2}} - \frac{w^2(l/2)}{4 \rho^2} \right) - \frac{w^2(l/2)}{4 \rho^2} \varphi_1 - \left(1 + \frac{EA}{k l} \right) \tau_{11} = 0 \quad (3)$$

Since

$$w(l/2) \sim \frac{C_1 l^2}{\pi^2}$$

Similar expressions can be derived for the other boundary conditions.

2.3.3.3 Axial Shortening

The initial axial shortening due to the application of temperature and lateral load is determined as a function of the polynomial coefficients expressing the lateral deflection. The axial shortening is determined as a function of the actual lateral deflection rather than by a function of the amplitude of the characteristic modes, in order to improve the accuracy of the solution (see Eq. 5 of Paragraph 2.3.2).

Assume that the lateral deflection (w_o) is readily available and is expressible as a polynomial of the fourth degree or less. Higher degrees of polynomials can be treated in a manner similar to that indicated below. Differentiating the deflection results in the slope (w_o') which can be squared and integrated over the length of the column to obtain twice the axial shortening

$$\text{i. e.} \quad w_o = a + b \xi + c \xi^2 + d \xi^3 + e \xi^4 \quad (1a)$$

$$w_o' = \frac{1}{l} \left[b + 2c \xi + 3d \xi^2 + 4e \xi^3 \right] \quad (1b)$$

$$2\Delta_o = \int_0^l (w_o')^2 dx = \frac{1}{l} \left[b^2 + 2bc + \frac{6bd + 4c^2}{3} + 2be + 3cd + \frac{16ce + 9d^2}{5} + 4de + \frac{16e^2}{7} \right] \quad (1c)$$

2.3.3.3 (Cont'd)

Typical examples for simple-simple columns are presented.

For Constant Thermal Gradient $\left(\frac{\Delta\alpha T}{h}\right)$

$$-\frac{\Delta\alpha T}{h} = \frac{1}{l^2} m_o \xi^0$$

$$j = 0 \text{ and } w_o = \frac{m_o}{(0+1)(0+2)} (\xi^{0+2} - \xi) \text{ from Eq. (3a) of Paragraph 2.3.3.1.1}$$

$$w_o = \frac{m_o}{2} (\xi^2 - \xi) = -\frac{l^2 \Delta\alpha T}{2h} (\xi^2 - \xi) = m_T (\xi - \xi^2)$$

where

$$m_T = \frac{l^2 \Delta\alpha T}{2h}$$

Substituting $b = m_T$ and $c = -m_T$ in Eq. (1c)

results in

$$2l \Delta_o = \frac{1}{3} m_T^2 \quad (2a)$$

For Uniform Lateral Load (q)

$$w_o = \frac{q l^4}{24EI} (\xi - 2\xi^3 + \xi^4) = m_q (\xi - 2\xi^3 + \xi^4)$$

$$\text{where } m_q = \frac{q l^4}{24EI}$$

substituting $b = m_q$, $d = -2m_q$, and $e = m_q$

$$\text{results in } 2l \Delta_o = .486 m_q^2 \quad (2b)$$

For Combined Uniform Load and Thermal Gradient

$$w_o = (m_T + m_q) \xi - m_T \xi^2 - 2m_q \xi^3 + m_q \xi^4 \quad (2c)$$

results in

$$2l \Delta_o = .333 m_T^2 + .800 m_T m_q + .486 m_q^2 \quad (2d)$$

2.3.3.3 (Cont'd)

It should be noted that any manufacturing or loading eccentricities will augment the lateral deflection parameters (C_i and d_i) by causing an axial shortening when the column is loaded, but should not be included directly in the Δ_0 (initial axial shortening because of thermal gradients and lateral loads) of the compatibility equation. The lateral eccentricities due to the various causes are additional and are included in the additional axial motion of the ends. The axial shortening however, is not linear but is proportional to the square of the deformations. Thus the initial axial shortening (Δ_0) must be computed as the difference of the axial shortening between the column including the manufactured eccentricities and the column containing only the manufactured eccentricities (w_{00}).

$$\text{i.e. } \Delta_0 = \int_0^L (w_q' + w_T' + w_{00}')^2 dx - \int_0^L (w_{00}')^2 dx \quad (3)$$

2.3.4 Spring Constant (k)

The value of k to be employed in the solution of the problem can significantly effect the magnitude of the resulting axial load. A column with zero axial restraint ($k = 0$) does not develop any axial load. The load increases with increasing axial restraint until it reaches a maximum for complete restraint ($k = \infty$).

The value of k is the value of the load produced by moving one end of the column a unit distance in the axial direction relative to the other end (assuming that the column offers no resistance). This is the stiffness coefficient defined in Section 4.2.5 of Reference 2-2. If both ends of the column are spring mounted, then the value of k is determined by putting the two end springs in series

$$\begin{aligned} \text{i.e. } \frac{1}{k} &= \frac{1}{k_1} + \frac{1}{k_2} \\ \therefore k &= \frac{k_1 k_2}{k_1 + k_2} \end{aligned} \quad (1)$$

where k = effective axial restraint

k_1, k_2 = axial restraint at the ends of the column

The method of determining the value of k in a composite structure is illustrated below for a truss joint.

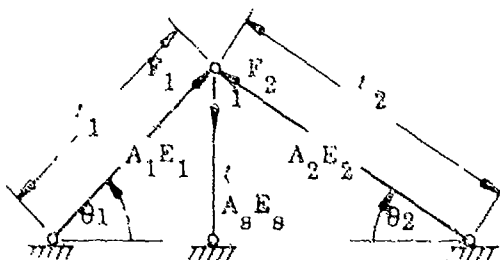


FIGURE 2.3.4-1 DETERMINATE TRUSS JOINT

The equilibrium equations are:

$$F_1 \sin \theta_1 + F_2 \sin \theta_2 = 1 \quad (2a)$$

$$F_1 \cos \theta_1 = F_2 \cos \theta_2 \quad (2b)$$

2.3.4 (Cont'd)

For symmetrical geometry ($\theta_1 = \theta_2 = \theta$; $A_1 E_1 = A_2 E_2 = AE$)

$$F_1 = F_2 = \frac{1}{2 \sin \theta} \quad (2c)$$

$$\frac{1}{k} = \Delta = \frac{F \ell / \sin \theta}{AE \sin \theta} = \frac{\ell}{2AE \sin^3 \theta} \quad (3a)$$

$$\frac{E_s A_s}{\ell} = \frac{1}{2 \sin^3 \theta} \left(\frac{E_s A_s}{EA} \right) \quad \text{where } E_s A_s = \text{axial stiffness of strut} \quad (4a)$$

For unsymmetrical geometry

The load will cause a non-vertical motion of center strut and change the angles. For small deformations the following relationships result:

$$F_1 = \frac{1}{\sin \theta_1 (1 + \cot \theta_1 \tan \theta_2)} \quad ; \quad F_2 = \frac{1}{\sin \theta_2 (1 + \cot \theta_2 \tan \theta_1)} \quad (2d)$$

$$\begin{aligned} \frac{1}{k} = \Delta &\sim \frac{1}{2} \left(\frac{F_1 \ell_1}{A_1 E_1 \sin \theta_1} + \frac{F_2 \ell_2}{A_2 E_2 \sin \theta_2} \right) \\ &= \frac{\ell}{2} \left(\frac{1/A_1 E_1}{\sin^3 \theta_1 (1 + \cot \theta_1 \tan \theta_2)} + \frac{1/A_2 E_2}{\sin^3 \theta_2 (1 + \cot \theta_2 \tan \theta_1)} \right) \end{aligned} \quad (3b)$$

$$\text{Thus } \frac{E_s A_s}{k \ell} \sim \frac{E_s A_s}{2} \left(\frac{1}{A_1 E_1 \sin^3 \theta_1 \left(1 + \frac{\tan \theta_2}{\tan \theta_1} \right)} + \frac{1}{A_2 E_2 \sin^3 \theta_2 \left(1 + \frac{\tan \theta_1}{\tan \theta_2} \right)} \right)$$

$$\text{This reduces to } \frac{E_s A_s}{k \ell} \sim \frac{1}{2 \sin^3 \theta} \left(\frac{E_s A_s}{EA} \right) \quad \text{for symmetrical geometry.}$$

2.3.5 Effect of Non-Linearity in Spring or Material

Solutions to the axial load in the column can be obtained even when the axial restraint and/or stiffness of the column varies with the axial load.

The solution of a column with a variable axial restraint can be obtained by superimposing a plot of the flexibility parameter $\left[EA/\ell k = EA/\ell k (\eta_1) \right]$ as a function of the axial load parameter (η_1) upon a plot of the solution of the compatibility equation for the load parameter $\left[\eta_1 = \eta_1 \left(\frac{EA}{\ell k} \right) \right]$ as a function of the flexibility parameter. The first plot is obtained directly from the spring characteristics whereas the second plot is obtained by varying the value of $EA/\ell k$ to obtain different values of η_1 . The intersection of the two plots, as illustrated in Figure 2.3.5-1, will result in a compatible solution.

2.3.5 (Cont'd)

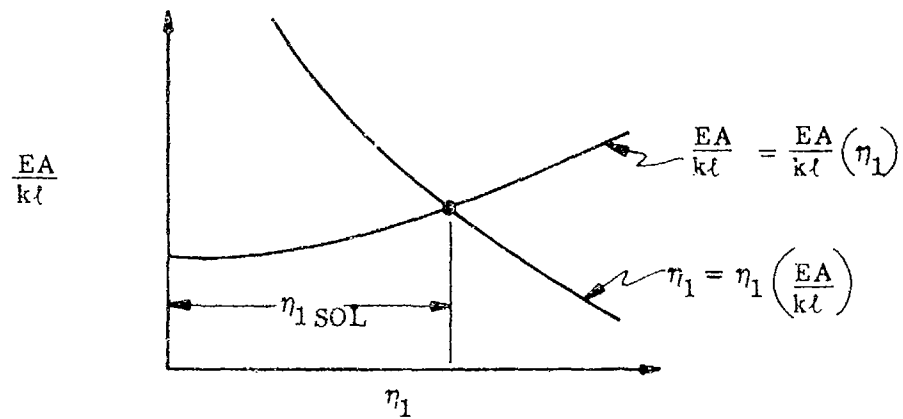


FIGURE 2.3.5-1 FLEXIBILITY VS AXIAL LOAD

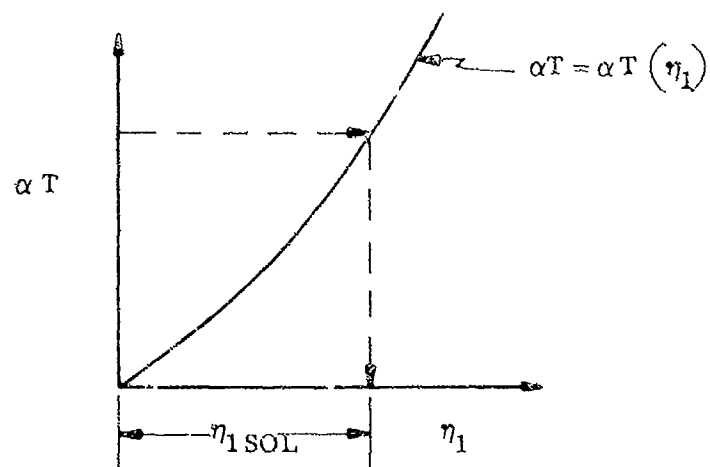


FIGURE 2.3.5-2 AXIAL LOAD VS AVERAGE THERMAL EXPANSION

2.3.5 (Cont'd)

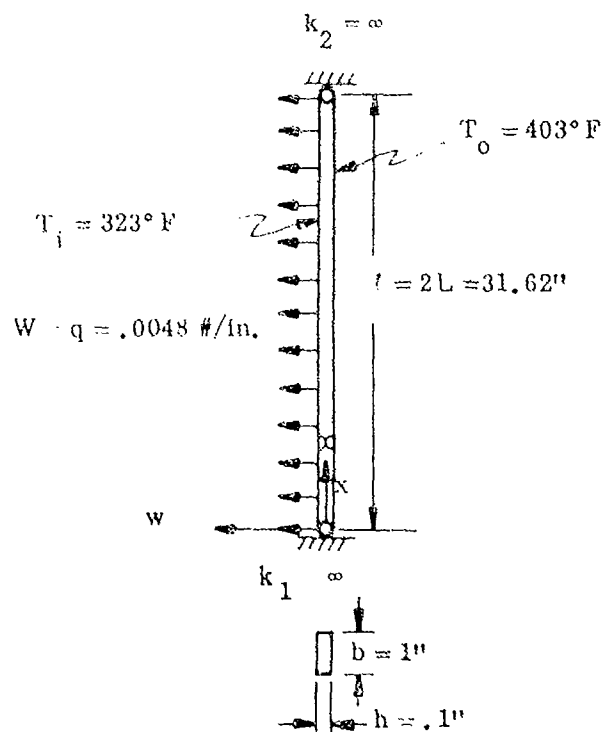
The solution of a column with a load dependent material stiffness (E_s) due to the plasticity of the material can similarly be obtained by a trial and error procedure. Assuming a value of η_1 will determine a value of E_s . This will determine a value of the flexibility parameter $\frac{E_s A}{k\ell}$ which varies linearly with E_s ; values of the lateral deflection parameters (C_i) whose lateral load component varies inversely with E_s and whose thermal component is unaffected; and the axial shortening parameters (d_i) which varies as the square of the C_i . Corresponding to these values of η_1 , values of \bar{d}_1 can be computed and employed in Figures 2.2.4 to obtain values of \bar{b} which can be utilized to calculate αT . The actual value of η_1 results when the calculated value of αT corresponds to the correct value of αT . A graphical approach is illustrated in Figure 2.3.5-2.

This above method can be utilized to solve a problem with a load-dependent axial restraint and a load dependent material stiffness. The above technique assumes that the axial and bending stiffness of the column is not significantly affected by the variation of the stress through the thickness of the cross section but is determined primarily by the mean stress (F/A).

2.3.6 Illustrative Problem

The computation procedure for the approximate solution is relatively simple and is illustrated for a problem which can be directly compared to an "exact" solution obtained from Table 1.5-1.

The temperature on each face is constant along the length with a linear gradient through the thickness.



$$\begin{aligned} k &= \infty \\ \ell &= 2L = 31.62 = 10 \sqrt{10} \\ h &= .1 \\ b &= 1 \\ \alpha &= 6(10)^{-6} \text{ in/in/}^\circ \text{F} \\ E &= 12(10)^6 \text{ pounds per square inch} \\ \Delta T &= T_i - T_o = -80^\circ \text{F} \\ T &= \frac{T_o + T_i}{2} = 363^\circ \text{F} \\ W &= q = .0048 \text{ \#/in.} \end{aligned}$$

2.3.6 (Cont'd)

AXIAL EXPANSION

From Eq. (1b) of Paragraph 2.2.4

$$\alpha T = \alpha \frac{T_o + T_i}{2} = 6(10)^{-6} \cdot 323 = .0021780$$

AXIAL SHORTENING

From Eq. (2d) of Paragraph 2.3.3.3

$$\frac{\Delta_o}{l} = \frac{1}{l^2} \left[\frac{1}{6} m_T^2 + .400 m_T m_q + .243 m_q^2 \right]$$

but
$$m_T = \frac{l^2 \Delta \alpha T}{2h} = \frac{1000 \cdot 6(10)^{-6} \cdot (-80)}{2(.1)} = -2.4''$$

and
$$m_q = \frac{q l^4}{24 EI} = \frac{+.0048 (10)^6}{(24)12(10)^6 \cdot \frac{1}{12} (.1)^3} = +.2''$$

$$\therefore \frac{\Delta_o}{l} = \frac{1}{1000} \left[\frac{1}{6} (2.4)^2 + .4(2.4)(-.2) + .243(.2)^2 \right] = 7.777 (10)^{-4}$$

From Eq. (2c) of Paragraph 2.2.4

$$\epsilon_1 = \lambda_1 \rho^2 = \lambda_1 \left(\frac{l}{A} \right)^2 = \frac{\pi^2}{l^2} \frac{h^2}{12} = \frac{\pi^2}{1000} \frac{(.1)^2}{12} = 8.225 (10)^{-6}$$

$$\therefore \frac{\alpha T}{\epsilon_1} = 264.8 \quad \text{and} \quad \frac{\Delta_o}{l \epsilon_1} = 94.56$$

EXPANSION OF CURVATURE

From Eq. (2c) of Paragraph 2.3.3.3

$$w_o = (m_T + m_q) \xi - m_T \xi^2 - 2m_q \xi^3 + m_q \xi^4 \quad (1a)$$

$$\therefore w_o' = \frac{1}{l} \left[m_T + m_q - 2m_T \xi - 6m_q \xi^2 + 4m_q \xi^3 \right] \quad (1b)$$

$$\therefore w_o'' = \frac{1}{l^2} \left[-2m_T - 12m_q \xi + 12m_q \xi^2 \right] \quad (1c)$$

2.3.6 (Cont'd)

Direct substitution in the Fourier sine expansion or the use of Table 2.3.3.1.2-1 will enable the determination of the magnitude of the curvatures. Direct substitution is employed in this illustration to obtain general equations for the case of uniform load and thermal gradients. The Fourier coefficients defined by Eq. (2b) of Paragraph 2.3.3.1.2 are

$$S(0, i) = \frac{1 - (-1)^i}{i\pi}$$

$$S(1, i) = -\frac{(-1)^i}{i\pi}$$

and

$$S(2, i) = \frac{-(-1)^i}{i\pi} + \frac{2}{(i\pi)^3} \left[(-1)^i - 1 \right]$$

From Eq. (2a) of Paragraph 2.3.3.1.2

$$\therefore C_i = \frac{2}{l^2} \sum_j \sum_i m_j S(j, i)$$

$$C_i = \frac{2}{l^2} \left[(-2 m_T) \left(\frac{1 - (-1)^i}{i\pi} \right) + (-12 m_q) \left(\frac{-(-1)^i}{i\pi} \right) \right. \\ \left. + (12 m_q) \left(\frac{-(-1)^i}{i\pi} + \frac{2}{(i\pi)^3} \left[(-1)^i - 1 \right] \right) \right]$$

$$C_i = \frac{2}{l^2} \left[(-2 m_T) \frac{(1 - (-1)^i)}{i\pi} - \frac{24 m_q}{(i\pi)^3} (1 - (-1)^i) \right]$$

$$C_i = -\frac{4}{l^2} (1 - (-1)^i) \left[\frac{m_T}{i\pi} + \frac{12 m_q}{(i\pi)^3} \right]$$

$$\therefore \text{ for } C_{i(\text{odd})} = \frac{-4 l^2 C_{2k+1}}{2} = \frac{4}{2k+1} m_T + \frac{(48/\pi^3) m_q}{(2k+1)^3} \quad (2a)$$

$$\text{and for } C_{i(\text{even})} \quad C_{2k} = 0 \quad (2b)$$

2.3.6 (Cont'd)

The values of C_i are then calculated. The same values would be obtained from Table

$$2.3.3.1.2-1 \left[\frac{C_i t^2}{2} = \sum \sum m_j S(j, i) \right]$$

$$\therefore -\frac{t^2 C_1}{2} \begin{cases} = 4/\pi (m_T) + 48/\pi^3 (m_Q) \\ = 1.27324(-2.4) + 1.548076(+.2) \\ = -3.05578 + .30962 = -2.746 \end{cases}$$

$$-\frac{t^2 C_3}{2} = -\frac{1}{3} (3.05578) + \frac{1}{9} (.30962) = -1.007$$

$$-\frac{t^2 C_5}{2} = -\frac{1}{5} (3.05578) + \frac{1}{25} (.30962) = -.609$$

$$-\frac{t^2 C_7}{2} = -\frac{1}{7} (3.05578) + \frac{1}{49} (.30962) = -.435$$

PERTINENT PARAMETERS

From Eqs. (2b) and (4b) of Paragraph 2.3.1

$$d_1 = \left(\frac{t^2 C_1}{2} \right)^2 \frac{1}{\pi^4} \rho^2 t^2 \quad \text{and} \quad \frac{d_1 \eta_1}{\eta_1} = \frac{d_1}{t^2}$$

$$\therefore d_1 = \frac{(2.746)^2}{\pi^4 \left(\frac{.1}{12} \right)^2} = 92.895$$

similarly

$$d_3 = 1.388 \quad \text{and} \quad \frac{d_3 \eta_3}{\eta_1} = d_3/3^2 = .154$$

$$d_5 = .183 \quad \frac{d_5 \eta_5}{\eta_1} = d_5/5^2 = .007$$

$$d_7 = .047 \quad \frac{d_7 \eta_7}{\eta_1} = d_7/7^2 = .001$$

2.3.6 (Cont'd)

$$\therefore \sum_{i=1}^8 d_i = d_2 + d_3 + d_5 + d_7 = 94.513$$

and

$$\frac{\sum_{i=2}^8 \eta_i d_i}{\eta_1} = d_3/3^2 + d_5/5^2 + d_7/7^2 = .162$$

Employing Eq. (9a) of Paragraph 2.24 and including correction for finite sum approximation of the initial shortening (Eq. (5) of Paragraph 2.3.2) results in

$$0 = \frac{\alpha T}{\epsilon_1} - \frac{\Delta_o}{\ell \epsilon_1} - \left(\frac{\Delta_o}{\ell \epsilon_1} - \sum_{i=1}^8 d_i \right) - d_1 \varphi_1 - \left(1 + \frac{EA}{k\ell} + 2 \sum_{i=2}^8 \frac{\eta_i d_i}{\eta_1} \right) \eta_1 \quad (3)$$

Substituting the calculated values results in

$$0 = (264.8 - 94.555) - (94.555 - 94.513) - 92.895 \varphi_1 - (1.234) \eta_1$$

$$0 = 170.203 - 92.895 \varphi_1 - (1.234) \eta_1$$

$$0 = 137.93 - 75.23 \varphi_1 - \eta_1 = \bar{b} - \bar{d}_1 \varphi_1 - \eta_1$$

Entering Figure 2.2.4-2 with $\bar{b} = 137.93$ and $\bar{d}_1 = 75.28$ results in the solution $\eta_1 = .407$.

AXIAL LOAD

From Eq. (3) of Paragraph 2.3.2

$$F = AE \epsilon_1 \eta_1 = (1.1) (1) (12) (10)^6 8.225 (10)^{-6} .407 = 4.017\#$$

LATERAL DEFLECTION

From Eq. (4a) of Paragraph 2.3.2

$$w_{\max} = w(\ell/2) = \sum \frac{w_{oi}}{1-\eta_i} = -\frac{\ell^2}{\pi^2} \sum \frac{C_i}{i^2(1-\eta_i)} = -\frac{2}{\pi^2} \sum \frac{\ell^2 C_i}{2i^2(1-\eta_i)}$$

2.3.6 (Cont'd)

$$\therefore w_{\max} \sim -\frac{2}{\pi^2} \left[\frac{-2.74616}{1-.407} + \frac{-1.00712}{9(1-.407/9)} + \frac{-.60867}{25(1-.407/25)} + \frac{-.43523}{49(1-.407/49)} \right]$$

$$w_{\max} \sim .97''$$

The geometry and loads employed in this illustrative problem results in parameters which correspond to an exact solution found in Table 1.5-1.

These parameters are

$$\bar{T}_D = \alpha \left(\frac{L}{h} \right)^2 (T_o - T_1) = 12.0$$

$$\bar{W} = \frac{12W}{Eb} \left(\frac{L}{h} \right)^4 = 3.0$$

$$\bar{T} = \alpha \left(\frac{L}{h} \right)^2 (T_o + T_1) = 108.9$$

A comparison of the exact and approximate solutions is presented below.

	Exact Solution	Approximate Solution
$\bar{\lambda}$	- 1.000	- 1.005
η_1	.405	.407
F	4.01#	4.02#
w_{\max}	.92''	.97''

Note that the approximate solution results in slightly larger values. The approximate method, however can be applied to more general types of loadings and boundary conditions.

It should be noted that the illustrated example shown in Figure 1.7-1 of Section 1 is solved by an interpolation of the "exact" tabular values. The approximate graphical procedure as well as the interpolation of tables procedure is subjected to relatively greater errors in the vicinity of low load ratios. This is because of the greater relative significance of the higher modes in the graphic approach and the large slope of the \bar{T} vs $\bar{\lambda}$ (as illustrated in Figure 1.5-2 of Section 1 of the tabular solution which permits large variations in $\bar{\lambda}$ for small variations in \bar{T}). Fortunately the design of the column is not determined by this condition of low axial load. The interpolation and graphical solutions are not subject to significant errors, however, when the load ratios are higher.

A comparison of the interpolation solution and of the approximate graphical solution of this report is presented as follows:

	Interpolated Sol.	Approx. Graphical Sol.
$\bar{\lambda}$	- .524	- .607
η_1	.111	.155
F	27.4	37.2
w_{\max}	.16	.18

2.4 REFERENCES

- 2-1. Switzky, H., "Approximate Solution For an Axially Restrained Column Subjected to Elevated Temperatures and Lateral Loads," Republic Aviation Corporation Report No. ARD-679-4, September 1961.
- 2-2. Switzky, H., Forray, M., and Newman, M. "Thermo-Structural Analysis Manual"-Volume I, Republic Aviation Corporation Report No. RAC 679-1, September 1960, revised November 1961 (to be published as WADD TR 60-517, Vol. I).

SECTION 3

APPROXIMATE SOLUTION FOR THE BUCKLING OF
ECCENTRIC COLUMNS

by

H. Switzky

SECTION 3
APPROXIMATE SOLUTION FOR THE BUCKLING OF
ECCENTRIC COLUMNS

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SECTION 3 - APPROXIMATE SOLUTION FOR THE BUCKLING OF ECCENTRIC COLUMNS

3.1 SUMMARY

An approximate solution $\left(\bar{F}_1 = \frac{K}{l^2} E_{So} I_o \left[1 - \sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{E_{To}}{E_{So}} \right) \right]^{2/3} \right)$ is obtained

for the buckling of an eccentric column which includes the effects of the stress level, the stress-strain relationship, and the initial eccentricity.

The analysis technique is presented in a nondimensional form to permit the analyst to consider various cross sections, boundary conditions, materials and degrees of eccentricity. The type of cross section is reflected in the I/A ratio which defines the square of the radius of gyration (r). The boundary conditions are contained in the value of K/l^2

which consider the end fixity and length of the column. The material characteristics are reflected in material parameters (E_A, σ_o, β) which represent the stress-strain relationship.

The nondimensional analysis graphs are presented for several values of the eccentricity

ratio $\left(\frac{a_{10}}{r} \right)$ which may occur because of initial waviness, eccentric loading, lateral loads, or thermal gradients. Analysis for intermediate values of the eccentricity can be conducted by interpolation.

Illustrative problems are presented to indicate the computation procedure and the effect of the initial eccentricity upon the stability of the structure.

3.1.1 Definition of Symbols

The following symbols are used throughout this section:

a_{10}	Original amplitude (eccentricity) of fundamental mode
h	Depth of cross section of the column
l	Length of column
m_j	Coefficient expressing initial deformations of the column as a power series
q	Lateral load acting on column
r	Radius of gyration of cross section
w	Lateral deflection of column
x	Axial coordinate of column
z'	Distance from reference axis
\bar{z}'	Distance from reference axis to bending axis
z_o'	Distance from reference axis to centroidal axis $\bar{z}_o' = \int z' dA / \int dA$
z	Distance from bending axis
A	Area of cross section
$C(j, 1)$	Fourier expansion coefficient $(C(j, 1) = \int_0^1 \xi^j \cos 2\pi \xi d\xi)$
$E = E_A$	Initial slope of the σ vs ϵ_σ curve
E_S	Secant modulus of the σ vs ϵ_σ curve $(E_S = \sigma/\epsilon_\sigma)$
E_{So}	Secant modulus at the bending axis
E_T	Tangent modulus, slope of the σ vs ϵ_σ curve $(E_T = d\sigma/d\epsilon_\sigma)$
E_{To}	Tangent modulus at the bending axis

3.1.1 (Cont'd)

\overline{EA}	Axial stiffness of cross section ($\overline{EA} = \int E_S dA$)
\overline{EI}	Bending stiffness of cross section ($\overline{EI} = \int E_S z^2 dA$)
$\overline{\overline{EI}}$	Buckling stiffness of cross section ($\overline{\overline{EI}} = \overline{EI} + \alpha \frac{\partial \overline{EI}}{\partial \alpha} = \overline{F}_1 / \frac{K}{l^2}$)
F	Axial load on structure (compression positive)
F_1	Buckling load of column with zero eccentricity or a linear material
\overline{F}_1	Buckling load of column ($\overline{F}_1 = \frac{K}{l^2} E_{So} I_o \left[1 - \left\{ \sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{E_{To}}{E_{So}} \right) \right\}^{2/3} \right]$)
F_E	Euler Buckling load ($F_E = \frac{K}{l^2} EI_o$)
I_o	Second moment of area (inertia) of cross section ($I_o = \int (z' - \bar{z}_o')^2 dA$)
K	Stability constant depending upon boundary conditions and the bending stiffness distribution along the columns ($K = l^2 \frac{\delta \alpha}{\delta w} = \frac{l^2 \int_0^l \phi (w_1'')^2 dx}{\int_0^l (w_1')^2 dx}$)
M	Moment acting on cross sections (Positive M causes compression in positive fibers).
Q_o	Third moment of area of cross section ($Q_o = \int (z' - \bar{z}_o')^3 dA$)
$S(j, 1)$	Fourier expansion coefficient ($S(j, 1) = \int_0^1 \xi^j \sin \pi \xi d\xi$)
T	Temperature rise above datum
α	Coefficient of linear thermal expansion
α	Parameter expressing variation of secant modulus in the cross section ($\alpha = \frac{\alpha}{\epsilon_o} \left(1 - \frac{E_{To} \gamma}{E_{So}} \right)$)
β	Nondimensional parameter employed in mathematical definition of stress-strain relationship (Section 3 of Reference 3-1)
γ	Nondimensional parameter expressing the average variation of the tangent modulus in the cross section ($\gamma = \frac{\alpha}{\epsilon_o} \int_0^1 E_T dz = \alpha E_{To}$)
$\Delta \alpha T$	Difference in thermal expansion
δ	Operator denoting a small variation
ϵ	Axial strain of cross section
ϵ_σ	Axial strain caused by stress ($\epsilon_\sigma = \epsilon - \alpha T$)
ϵ_1	Average axial strain corresponding to an axial load F_1
ϵ_o	Axial strain at the bending axis
$\overline{\epsilon}_1$	Average axial strain corresponding to an axial load \overline{F}_1
η	Shift of bending axis ($\eta = \bar{z}_o' - \bar{z}'$)
κ	Curvature
ξ	Nondimensional axial coordinate ($\xi = x/L$)
σ	Axial stress of cross section (compression positive)

3.1.1 (Cont'd)

σ_o	Axial stress at bending axis
σ_o	Reference stress in nondimensional stress-strain relationship (Section 3 of Reference 3-1)
ϕ	Nondimensional parameter expressing the variation of the bending stiffness along the length of the column ($\phi(x) = \bar{EI}/E_o I_o$)
Φ	Nondimensional parameter $\left(\Phi = \left(\frac{EA}{\sigma_o} \right) \left(\frac{KI}{Ab^2} \right) = \frac{EA \epsilon_1}{\sigma_o} \right)$
$\bar{\Phi}$	Nondimensional parameter $\bar{\Phi} = \left(\frac{EA}{\sigma_o} \right) \left(\frac{KI}{Ab^2} \right) \left[1 - \left\{ \sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{\bar{E}_T}{E_1} \right) \right\}^{2/3} \right]$ $= \frac{EA \bar{\epsilon}_1}{\sigma_o}$

SUBSCRIPTS

M	Caused by mechanical load
T	Caused by temperature
1	Pertaining to the first (fundamental) mode
o	Pertaining to initial or datum

3.1.2 Discussion of the Problem

The stability of a structure is a very complex problem. Exact solutions exist only for very few special cases. Approximations must be attempted when the solution is complicated by variations (reductions) of the stiffness with the applied load because of the resulting non-linearity of the equation. The reduction in the stiffness is caused by a non-linear structural material. The non-linear stress strain relationship reduces the stiffness of the structure by lowering the secant modulus and shifting the neutral axis.

Temperature, time, and the eccentricities of the structure tend to reduce the allowable magnitude of the applied load on the structure. Elevated temperatures reduce the stiffness of the structure, increase the non-linearity of the material because of plasticity at the stresses caused by the applied and thermal stresses, and usually increase the eccentricity of the structure. The axial forces acting through the eccentricity of the structure imposes moments upon the structure which increase at a greater rate than the applied load. These moments cause a variation of stresses through the cross sections which can reduce the bending stiffness. The larger the eccentricity the earlier the initiation of the non-linearity of the material and the lower the stability of the structure. The bending stiffness can also decrease with time because of creep of the material.

The approach, employed in this section, to examine the stability of a column (Figure 3.1.2-1) is to determine when the increase of external moment acting on the column cross sections tend to become greater than the increase of internal moment that can be generated by the cross sections. The axial load acting on the column at this time is the buckling load and is a function of the boundary conditions, the bending stiffness of the cross sections, and the rate that the bending stiffness is decreasing. The bending stiffness is a function of the cross-section, the magnitude of the stresses and their distribution and the stress-strain relationship. The bending stiffness usually decreases with an increase in the applied load, temperature, time, or eccentricity of the column. Thus the column becomes unstable when the applied load becomes equal to the buckling load which is determined by the bending stiffness distribution and its rate of decrease. This can occur by increasing the applied load and/or decreasing the magnitude of the buckling load. The buckling load is reduced by the eccentricity, the temperature, and time as indicated previously.

3.1.2 (Cont'd)

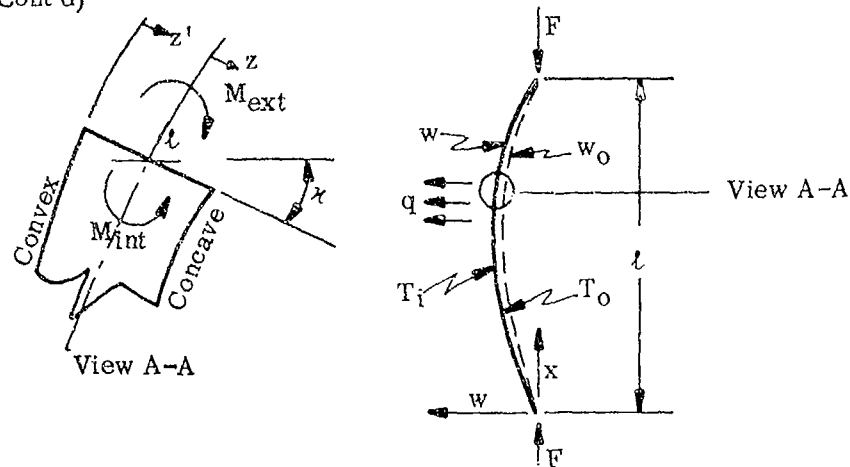


FIGURE 3.1.2-1 ECCENTRIC COLUMN SUBJECTED TO AXIAL LOADS

The evaluation of the buckling load requires the determination of the stiffness of the column and its rate of change. This is complicated by the shifting of the bending axis towards the convex side of the cross section because the higher stress and strain levels on the concave side reduce the secant modulus. The stiffness relationships, derived in Appendix A of Reference 3-2, are employed with the stability criterion described previously to obtain an approximate solution for the buckling load of the column in terms of the stress and lateral deflections. The solution is then approximated in terms of average stress level and initial eccentricity which can be readily computed. The initial eccentricities may result from the unloaded shape, eccentric loading, lateral loads, or thermal gradients. Methods of computing the initial eccentricity due to load and temperature are presented in Section 2 and summarized in this section.

An analogous approach can be employed to transform the loads or stresses to strains. Thus the column becomes unstable when the average axial strain becomes equal to a buckling strain. This approach is most convenient for a nondimensional presentation of the analysis.

The solution is predicated upon the following assumptions:

- (1) The bending strains are small compared to the axial strains in the column.
- (2) The temperature distribution can result in deformations and reductions in the moduli but do not cause significant thermal stresses.
- (3) There is a one-to-one relationship between the stress (σ) and the strain (ϵ_σ) for each fiber of the column. This assumes that there is no stress reversal throughout the loading history. It is also assumed that the stress-strain relationship can be approximated to a sufficient degree of accuracy by the relationship
$$\frac{E_A \epsilon}{\sigma_0} = (1-\beta) \frac{\sigma}{\sigma_0} + \beta \sinh \frac{\sigma}{\sigma_0}$$
 defined and described in Section 3 of Reference 3-1.

The approximate solution, which results from these assumptions, is of a form which does not violate known solutions of special cases and is indicative of the structural behavior in experiments.

3.2 ANALYSIS

A column subjected to loads and temperature (Figure 3.1.2-1) must satisfy equilibrium. Thus the change in applied moment (M_{ext}) must be equal to the change in the internal moment (M_{int}). The changes in external moment arise from changes in the axial load (F) and the lateral deflection (w). (The effect of elastic supports in introducing changes in the moment is reflected in the value of the stability constant K). The change in the internal moment is evidenced by a change in the curvature (κ_M) and a possible change in the bending stiffness (EI).

Referring to Figure 3.1.2-1, we note the equilibrium requirements.

$$M_{ext} + M_{int} = 0 \quad (1a)$$

and

$$-\delta(M_{ext}) = \delta(M_{int}) \quad (1b)$$

evaluating the change in moments, we obtain

$$-\delta(M_{ext}) = -\delta(Fw) = -[F(\delta w) + w(\delta F)] \quad (1c)$$

and

$$\delta(M_{int}) = \delta(EI \kappa_M) = [EI(\delta \kappa_M) + \kappa_M(\delta EI)] \quad (1d)$$

$$\therefore -[F(\delta w) + w(\delta F)] = [EI(\delta \kappa_M) + \kappa_M(\delta EI)] \quad (1e)$$

If the load is monotonically increasing, then the column will fail when the buckling load (\bar{F}_1) is applied. No additional incremental load can be applied so that the buckling load is defined by setting δF equal to zero. The above buckling criterion is equivalent to finding the load on a structure for which the deflection is undefined (Figure 9.1-1b of Reference 3-1). It is also equivalent to determining the load on the structure at which the ratio of $\delta F/\delta w$ is zero, where δF is the incremental load necessary to cause an incremental deflection (δw). The latter buckling criteria ($\delta F/\delta w = 0$) can also be employed to determine the stability of a column which exhibits creep.

Setting $\delta F = 0$ and $F = F_1$ in equation (1e) results in

$$-[\bar{F}_1 \delta w + w(0)] = EI \delta \kappa_M + \kappa_M \delta EI \quad (2a)$$

$$\therefore \bar{F}_1 = - \left[EI \frac{\delta \kappa_M}{\delta w} + \kappa_M \frac{\delta EI}{\delta w} \right] = - \frac{\delta \kappa_M}{\delta w} \left(EI + \kappa_M \frac{\delta EI}{\delta \kappa_M} \right)$$

$$\bar{F}_1 = - \frac{\delta \kappa_M}{\delta w} \bar{EI} \quad (2b)$$

Thus the buckling load is a function of the ratio of the change of curvature to the change of lateral deflection and of the bending stiffness modified by an expression indicating the rate of change of bending stiffness with curvature.

The first expression $\delta \kappa_M/\delta w$ is relatively simple to calculate if we assume that the deformation modes of the given structure do not change as the axial load is increased to the buckling load. Thus it is assumed that a pin ended column of constant bending stiff-

3.2 (Cont'd)

ness, which buckles elastically in a sine wave, will buckle in a sine wave even if it becomes plastic. This condition is satisfied as long as the ratio of the bending stiffness distribution $\left(\varphi(x) = \frac{\overline{EI}(x)}{E_0 I_0}\right)$ along the column does not change significantly with the axial load. The stiffness can change due to plasticity but it is assumed that the φ ratio does not change significantly. This is consistent with our assumption that the bending stresses are small compared to the axial stresses so that the material moduli remain approximately equal (for constant cross-section area). Thus the value of $\delta x_M / \delta w$ is assumed equal to the value of the "linear" column. This is the classical stability coefficient found in various textbooks which can be expressed as the curvature to deflection ratio as indicated above or by the general expression of Eq. (1a) of Paragraph 2.3.1.

$$-\frac{\delta x_M}{\delta w} = \frac{F_1}{\overline{EI}} = \lambda_1^2 = \frac{\int_0^l \varphi (w_1'')^2 dx}{\int_0^l (w_1')^2 dx} = \frac{K}{l^2} \quad (3)$$

Values of K can be found in various texts (e.g. References 3-3, and 3-4).

As an example for a pin-ended column of constant EI (i.e. $\varphi = 1$)

$$-\frac{\delta x_M}{\delta w} = - \frac{\delta \left(\frac{\pi^2}{l^2} \sin \frac{\pi x}{l} \right)}{\delta \left(\sin \frac{\pi x}{l} \right)} = \frac{\pi^2}{l^2} = \frac{K}{l^2} \quad (4a)$$

$$\text{or } \lambda_1^2 = \frac{\int_0^l (1) \left(-\frac{\pi^2}{l^2} \sin \frac{\pi x}{l} \right)^2 dx}{\int_0^l \left(\frac{\pi}{l} \cos \frac{\pi x}{l} \right)^2 dx} = \frac{\pi^2}{l^2} = \frac{K}{l^2} \quad (4b)$$

$$\therefore K = \pi^2 \quad (4c)$$

The second expression $\left(\overline{EI} = \overline{EI} + x_M \frac{\delta \overline{EI}}{\delta x_M} \right)$ is evaluated by utilizing the definition of the bending stiffness derived in Appendix A of Reference 3-2 which approximates the effects of the bending stress.

From Eq. (A10b) of Reference 3-2

$$\overline{EI} \approx E_{S0} I_0 (1 - 2\alpha^2 r^2) \quad (5a)$$

where

$$\alpha = \frac{x_M}{\epsilon_0} \left(1 - \frac{r_0}{E_{S0}} \gamma \right) \quad (5b)$$

3.2 (Cont'd)

and
$$\gamma = \int_0^z E_T dz / z E_T . \quad (5c)$$

Substituting Eq. (3), (5a), and (5b) in Eq. (2b) results in

$$\bar{F}_1 = \frac{K}{l^2} E_{So} I_o (1 - 6\alpha^2 r^2) \quad (6)$$

The solution for the buckling load is now expressed in terms of geometry and boundary conditions (l, I_o, r, K), the stress level at the neutral axis (E_{So}, E_{To}) and the stress distribution

$$\left(\alpha = \frac{\kappa_M}{\epsilon_o} \left(1 - \frac{E_{To}}{E_{So}} \gamma \right) \right). \text{ Unfortunately the stress distribution parameter } \alpha \text{ is not speci-}$$

fically defined but varies in a complex way with the applied load. It varies directly with the curvature (κ_M) which increases nonlinearly with the load, inversely with the axial strain (ϵ_o) which is defined by the stress-strain relationship (Figure 3.2-1) and directly with the

stress distribution factor $\left(1 - \frac{E_{To}}{E_{So}} \gamma \right)$ which increase with load and curvature. Some ad-

ditional approximations are required to reduce the solution to a simple form expressible in terms of the initial conditions. Each of the approximations employed introduces small over- or under-estimates of the buckling load. It is hoped that the combined effect of all the approximations will result in a reasonably accurate solution that considers the effects of eccentricity.

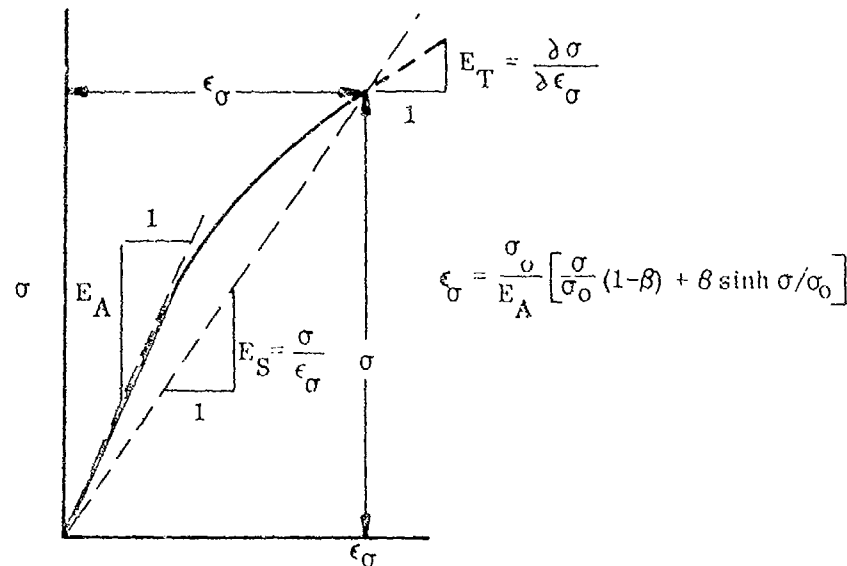


FIGURE 3.2-1 STRESS-STRAIN RELATIONSHIP

The first approximation is to assume that the lateral deflection will increase as a rectangular hyperbola of the form

$$w = \frac{a_{10} w_1}{1 - F/F_1} \quad (7)$$

3.2 (Cont'd)

where w = the lateral deflection
 w_1 = fundamental deflection mode
 a_{10} = initial amplitude of the fundamental mode

(The higher modes need not be considered in the buckling load problem since the column will buckle in the fundamental mode)

F_1 = buckling load for a linear or straight column

where
$$F_1 = \frac{K}{L^2} E_{So} I_o \quad (8)$$

The second approximation is to assume that γ has the magnitude of unity in evaluating the value of α . Under the assumption of small bending stresses, the magnitude of γ should be very close to but slightly lower than 1.

The third approximation is to assume the average axial strain (ϵ_o) to be equal to the value of $F_1/E_{So} A$ at buckling.

The above three approximations are consistent with the assumption of relatively small bending strains and result in a slight underestimation of α and a resulting slight overestimate of \bar{F}_1 .

The final approximation is to evaluate α with the stress distribution which exists at the cross section with the maximum lateral deflection. This assumption is made because the stability is primarily a function of the square of the curvature (see Eq. (3)) which is largest in the vicinity of the maximum lateral deflection (for columns of constant cross sections). This approximation of the bending to axial stress ratio overestimates α and results in an underestimate of \bar{F}_1 and compensates, to some degree, for the overestimates resulting from the small bending approximations.

The approximations result in the following equation

$$\bar{F}_1 = \frac{K}{L^2} E_{So} I_o \left[1 - \left\{ \sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{E_{To}}{E_{So}} \right) \right\}^{2/3} \right] \quad (9)$$

These approximations cannot be rigorously justified but they result in an expression for the stability of the column that can be simply applied to the analysis of a very complex problem and which retains the correct sense for the effect of the parameters. A well controlled test program is recommended to evaluate the accuracy of the resulting formulation and to supply empirical correction factors, as needed.

An examination of the experimental data of Reference 3-5 was conducted in Appendix C of Reference 3-2 and indicated good agreement with the results for specimens tested with eccentricities caused by eccentric loads or thermal gradients. The predicted results were slightly unconservative. Better agreement can possibly be obtained between the experimental results and the theoretical predictions by statistically determining the best value for the expression $(\sqrt{6} a_{10}/r)$. The graphs (Figures 3.3.1-1 through -10) can still be employed by modifying the indicated value of a_{10}/r . Additional test data is necessary, however, to examine the reliability of the above approximations for greater eccentricities and larger buckling stresses than those reported in Reference 3-5.

3.2 (Cont'd)

Using Eq. (9), the average buckling stress becomes

$$\bar{\sigma}_1 = \frac{\bar{F}_1}{A} = E_{So} \frac{Kr^2}{t^2} \left[1 - \left\{ \sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{E_{To}}{E_{So}} \right) \right\}^{2/3} \right] \quad (10)$$

and the average buckling strain becomes

$$\bar{\epsilon}_1 = \frac{\bar{\sigma}_1}{E_{So}} = \frac{Kr^2}{t^2} \left[1 - \left\{ \sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{E_{To}}{E_{So}} \right) \right\}^{2/3} \right] \quad (11a)$$

Since $\epsilon_1 = \frac{\sigma_1}{E_{So}} = \frac{F_1/A}{E_{So}} = \frac{K}{t^2} \frac{E_{So} I_o}{E_{So} A} = K \frac{r^2}{t^2}$ (11b)

$$\therefore \frac{\bar{\epsilon}_1}{\epsilon_1} = \left[1 - \left\{ \sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{E_{To}}{E_{So}} \right) \right\}^{2/3} \right] \quad (11c)$$

Letting $\bar{\Phi} = \frac{E_A}{\sigma_o} \bar{\epsilon}_1$ and $\Phi = \frac{E_A}{\sigma_o} \epsilon_1$

We obtain from Eqs. (9), (10), and (11c)

$$\frac{\bar{F}_1}{F_1} = \frac{\bar{\sigma}_1}{\sigma_1} = \frac{\bar{\epsilon}_1}{\epsilon_1} = \frac{\bar{\Phi}}{\Phi} = \left[1 - \left\{ \sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{E_{To}}{E_{So}} \right) \right\}^{2/3} \right] \quad (11d)$$

The factor $\left[1 - \left\{ \sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{E_{To}}{E_{So}} \right) \right\}^{2/3} \right]$ approximates the effect of the ec-

centricity and stress distribution. The effect is more pronounced with larger initial eccentricities or higher average stresses. It should be noted at this time that the factor was determined for sections which were symmetrical about the centroidal axis parallel to the bending axis ($Q_o = 0$). The stability of the column should decrease slightly with increasing values of Q_o/I_o as indicated by a comparison of the bending stiffness in the plastic range for $Q_o \neq 0$ with $Q_o = 0$ (see Eqs. (A10a) and (A10b) of Reference 3-2).

3.3 TECHNIQUE

The final formulation for the approximate solution for the buckling of an eccentric column is presented by the non-linear Eqs. (9), (10), or (11) of Sub-section 3.2. It is necessary to resort to graphical solutions in order to solve these equations. In order to simplify the analysis, a nondimensional solution is presented in Figure 3.3.1-1 to -10. The graphs are based upon the stress-strain relationship presented in Section 3 of Reference 3-1.

3.3.1 Nondimensional Buckling Curves

Equation (10) of Sub-section 3.2 can be transformed to

$$\left(\frac{\bar{\sigma}_1}{\sigma_o} \right) = \frac{1}{\frac{E_s}{E_A} \left[1 - \left\{ \sqrt{6} \left(\frac{a_{10}}{r} \right) \left(1 - \frac{E_{To}}{E_{So}} \right) \right\}^{2/3} \right]} \quad (1)$$

$$= \frac{E_A}{\sigma_o} \frac{K r^2}{t^2} = \frac{E_A}{\sigma_o} \left(\frac{K I}{A t^2} \right) = \Phi$$

Utilizing the stress-strain relationship of Section 3 of Reference 3-1 $\left(\frac{E_A \epsilon}{\sigma_o} = (1 - \beta) \frac{\sigma}{\sigma_o} + \beta \sinh \sigma / \sigma_o \right)$, it is possible to calculate values of E_s/E_A and E_T/E_s for various values of σ/σ_o and given in values of β . For a given value of the eccentricity ratio (a_{10}/r) it is then possible to calculate the value of Φ , which would result in a buckling stress ratio ($\bar{\sigma}_1/\sigma_o$), by direct substitution in Eq. (1). This relationship is plotted in Figures 3.3.1-1 to -10 for a large range of eccentricity ratios. The argument Φ is a function of the material properties (E_A/σ_o) and the geometry and boundary conditions (KI/At^2). The shape of the stress-strain relationship is represented by the value of β . The method of obtaining the material parameters from a uniaxial test is presented in Section 3 of Reference 3-1. The determination of the initial eccentricity which may be caused by initial waviness, eccentric loads, lateral loading, and thermal gradients is discussed in Paragraph 3.3.2.

The graphs for small eccentricity ratios should result in fairly good agreement with experimental data since the approximations employed should be satisfactory. The graphs for large eccentricity ratios are less accurate and are expected to be unconservative.

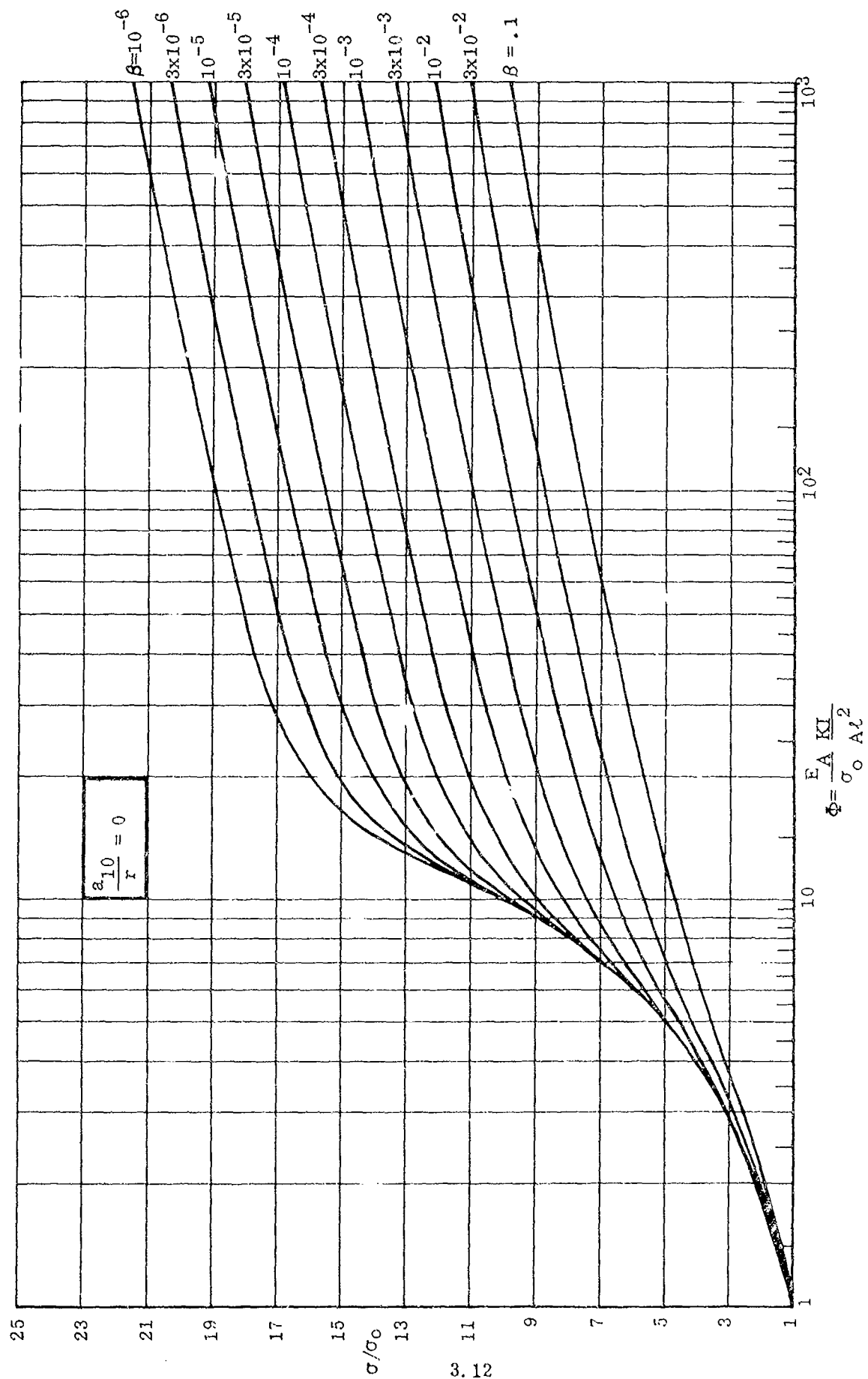


FIGURE 3.3.1-1 NON-DIMENSIONAL BUCKLING OF ECCENTRIC COLUMNS

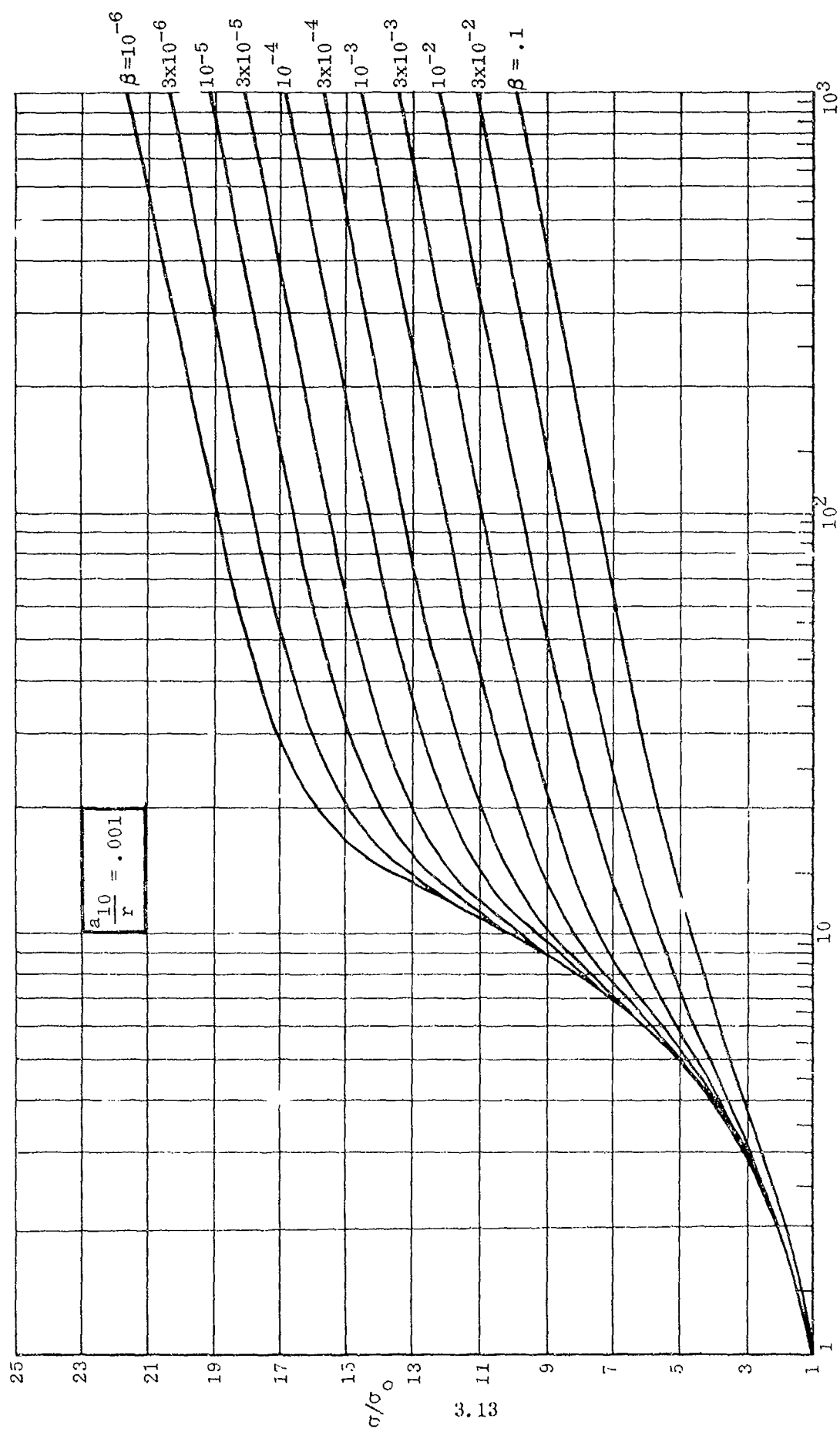


FIGURE 3.3.1-2 BUCKLING OF ECCENTRIC COLUMNS

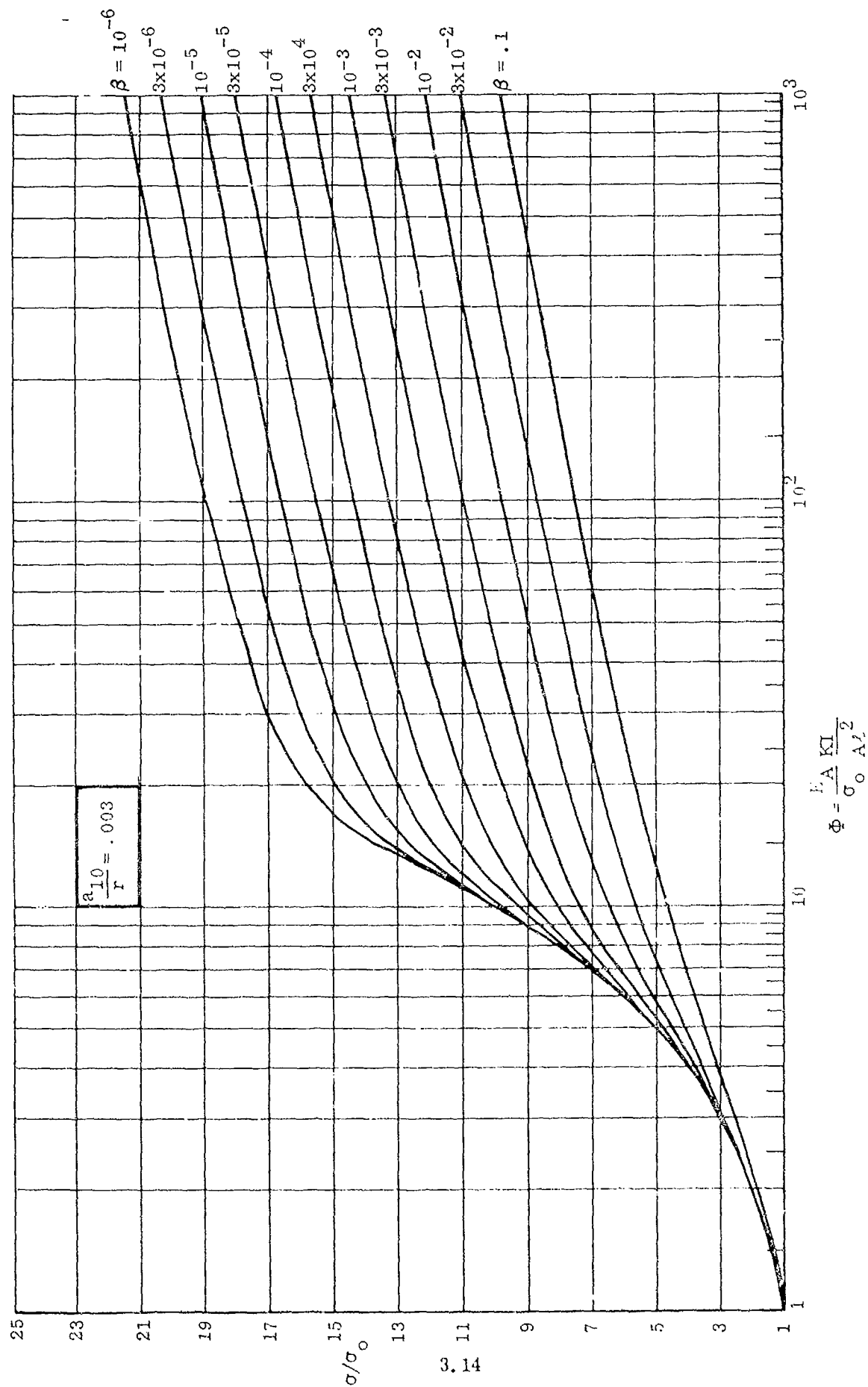


FIGURE 3.3.1-3 BUCKLING OF ECCENTRIC COLUMNS

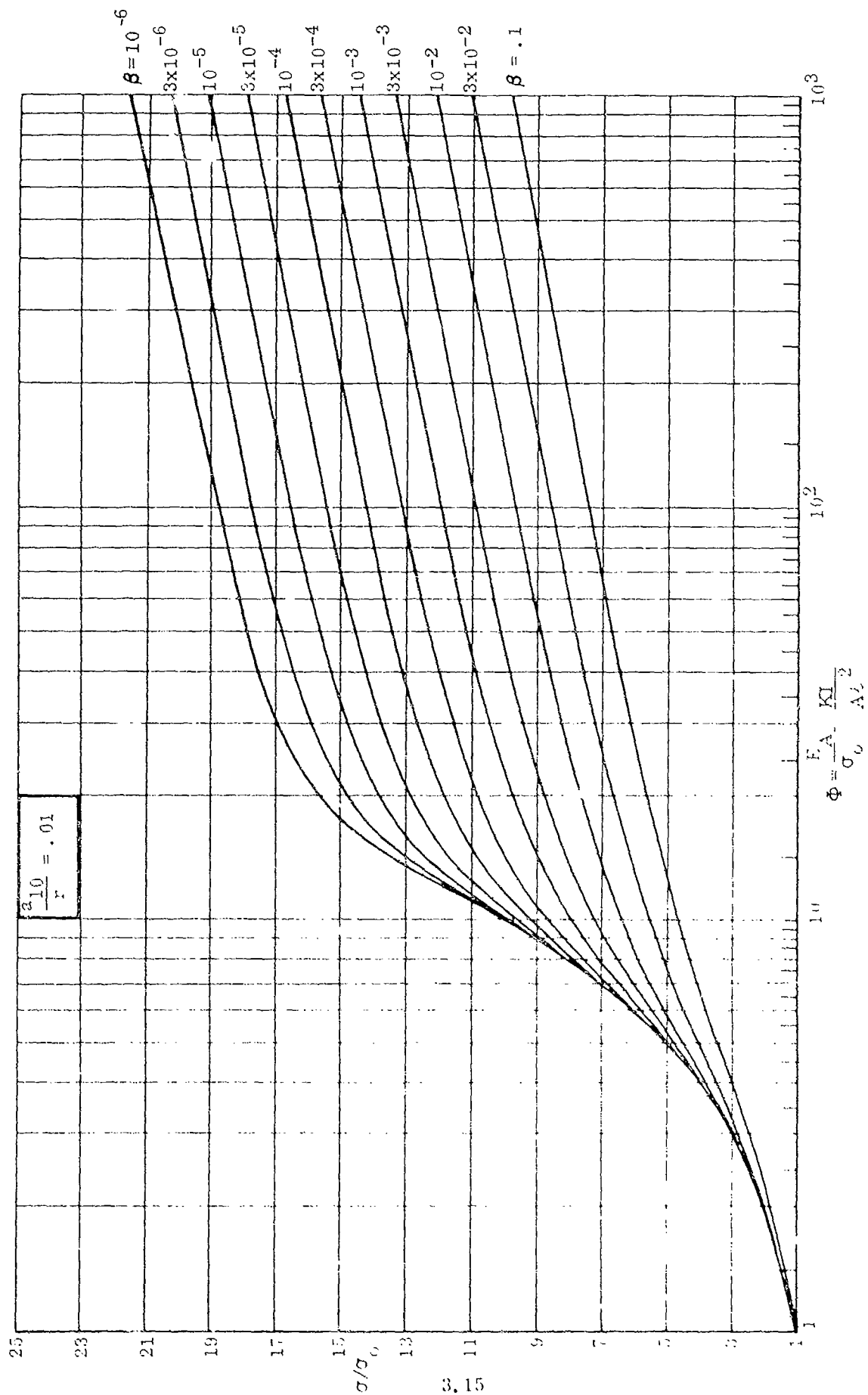


FIGURE 3.3.1-4 BUCKLING OF ECCENTRIC COLUMNS

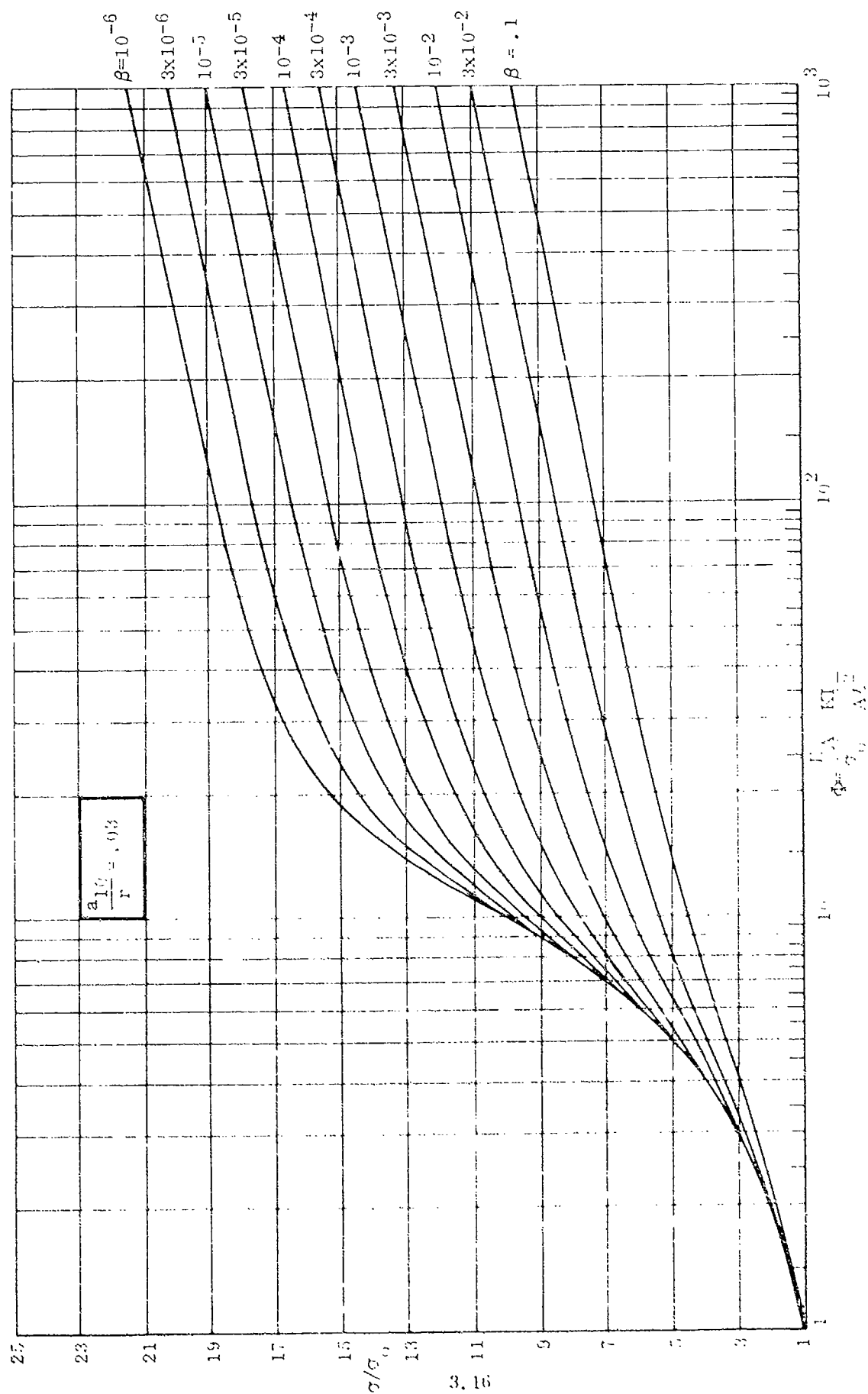


FIGURE 3.3.1-5 BUCKLING OF ECCENTRIC COLUMNS

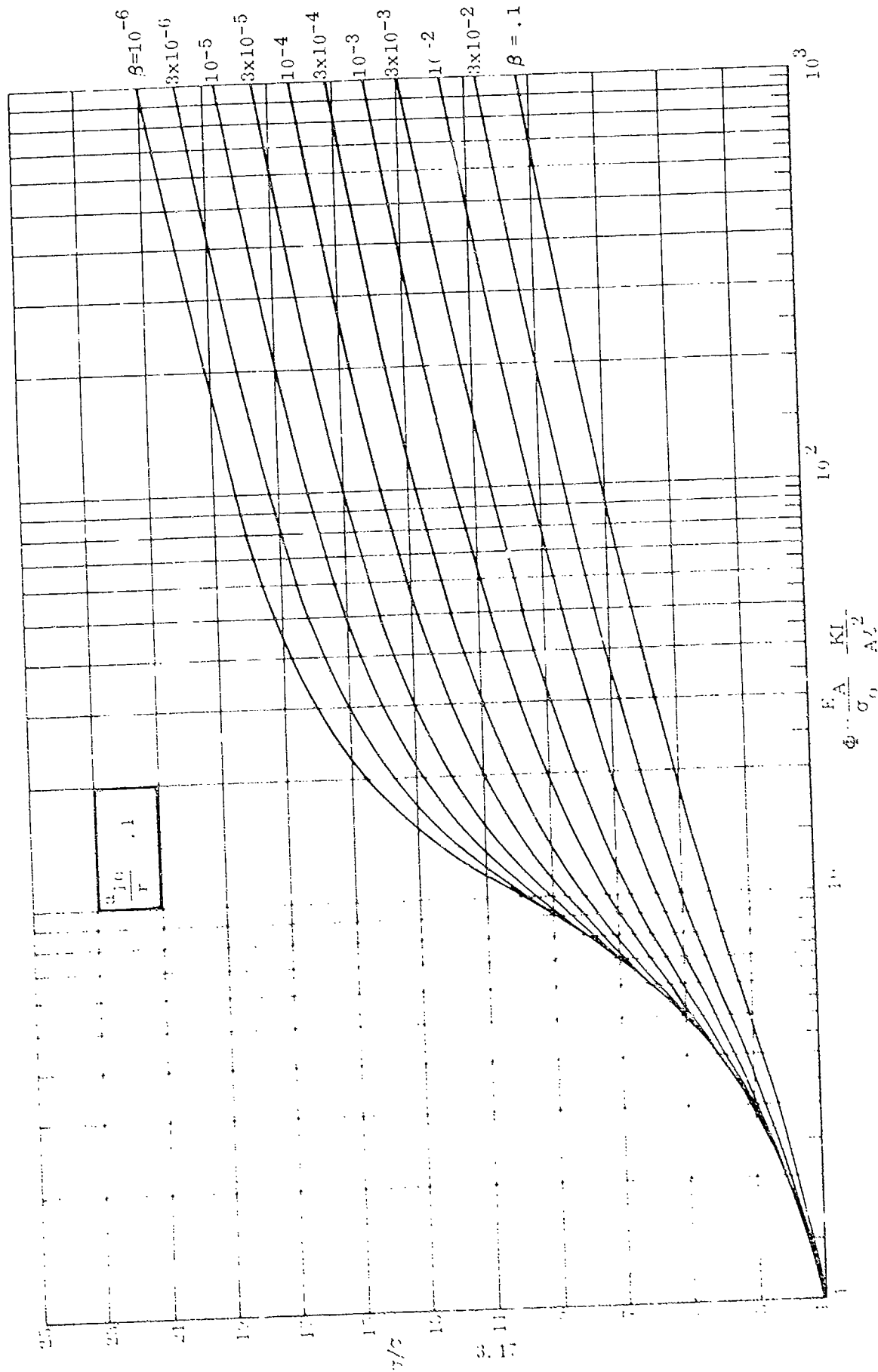


FIGURE 3.3.1-6 BUCKLING OF ECCENTRIC COLUMNS

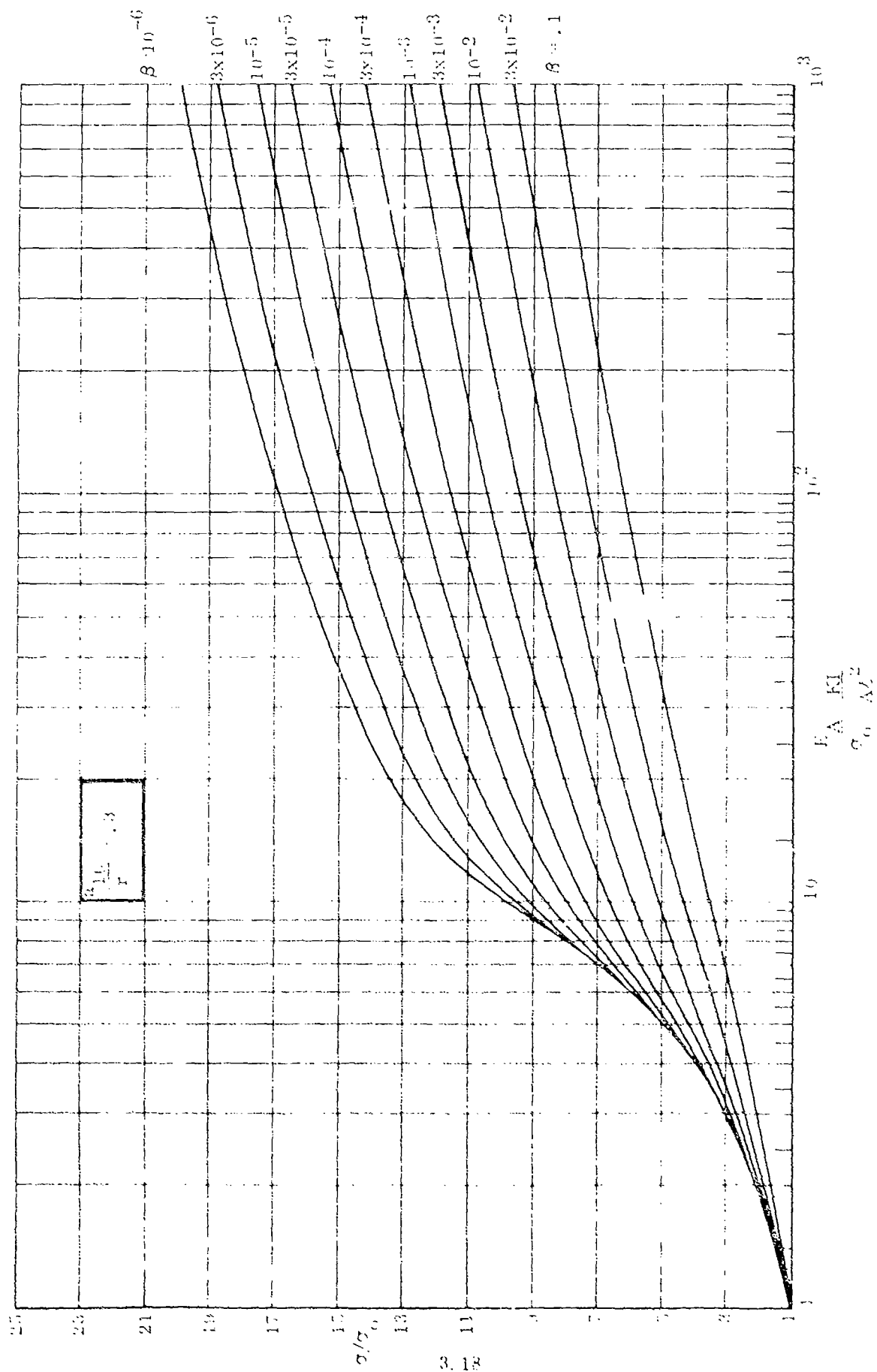


FIGURE 3.6.1.7 BUCKLING OF ECCENTRIC COLUMNS

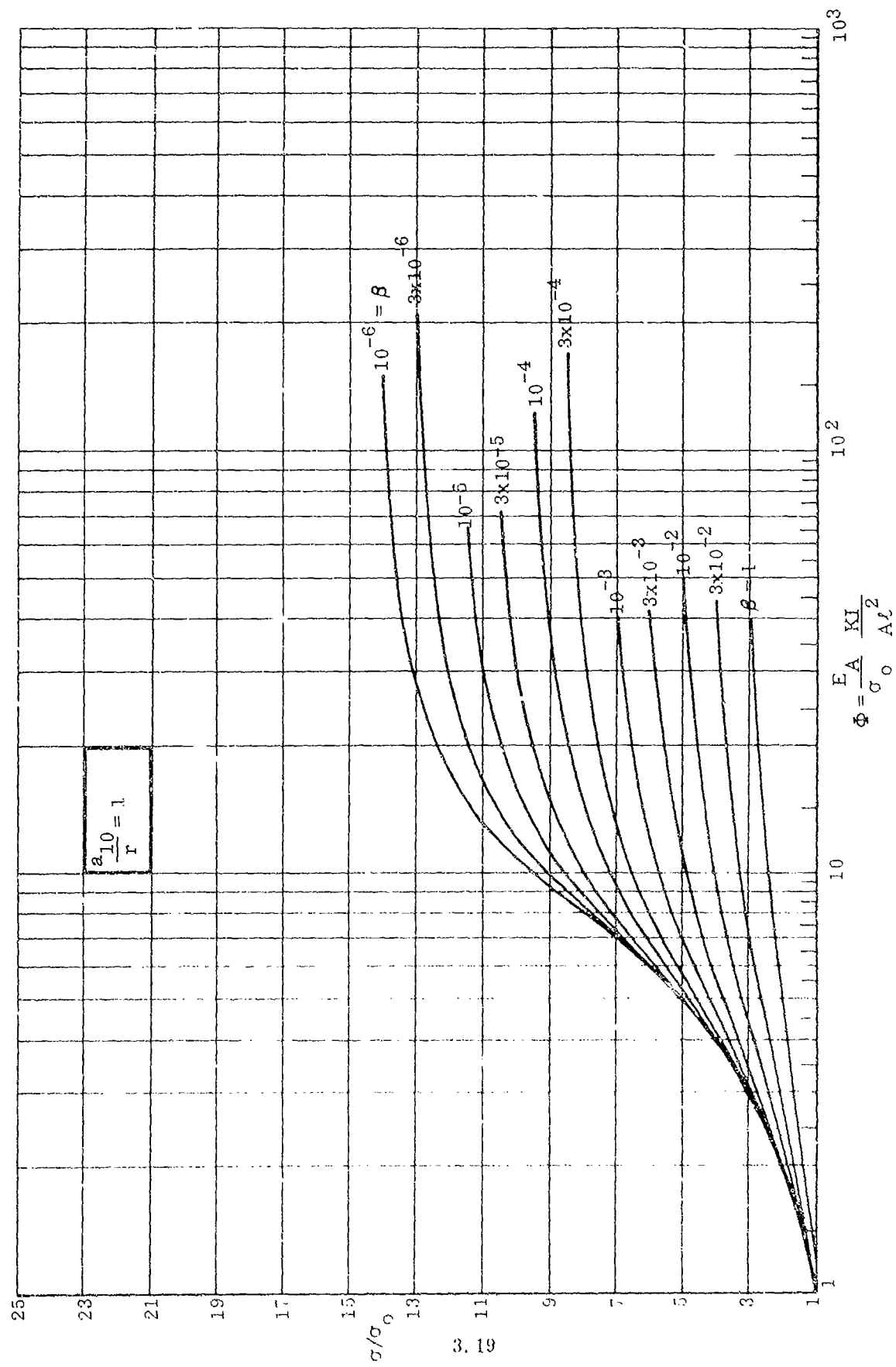


FIGURE 3.3.1-8 BUCKLING OF ECCENTRIC COLUMNS

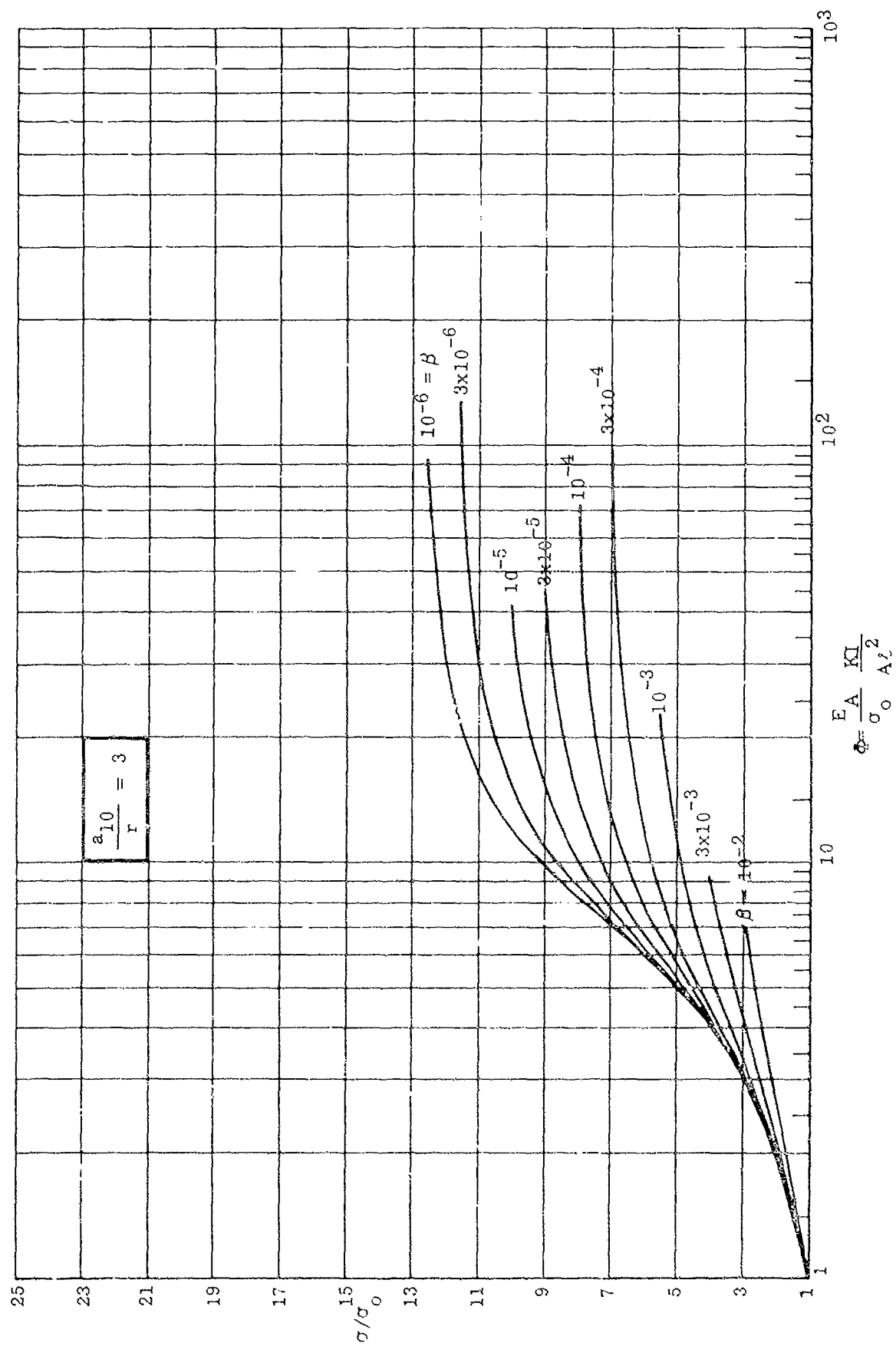


FIGURE 3.3.1-9 BUCKLING OF ECCENTRIC COLUMNS

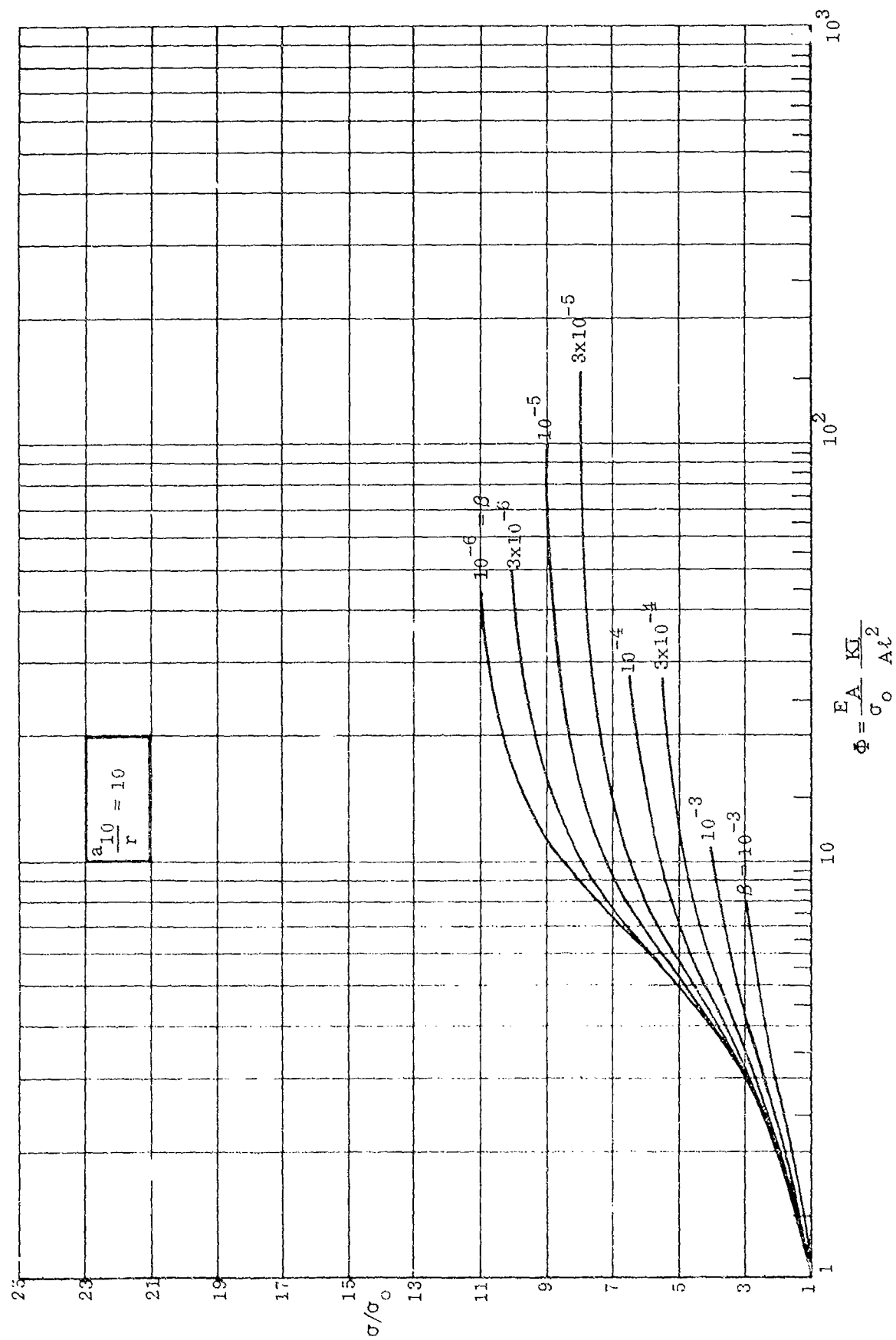


FIGURE 3.3.1-10 BUCKLING OF ECCENTRIC COLUMNS

3.3.2 Initial Eccentricity

The initial eccentricity must be evaluated in order to use the analysis curves.

Paragraph 2.3.3 indicated that the amplitude of the fundamental modes can be obtained by a weighted integration of the actual lateral deflection or by matching the deformation at discrete points.

Assuming that it is possible to express the initial eccentricity (w_0 due to shape, axial load location, lateral loads and temperatures) as an analytical expression, it is possible to determine the magnitude of the first mode by utilizing the orthogonality between the lateral deflection and curvature (Eq. (2) of Paragraph 2.2.1),

Thus, if

$$w_0 = \sum a_{10} w_i \quad (1a)$$

then from Eq. (2) of Paragraph 2.2.1 we obtain for constant bending stiffness ($C=1$)

$$a_{10} = \frac{\int_0^l w_0 \kappa_i dx}{\int_0^l w_i \kappa_i dx} \quad (1b)$$

The following formulae are applicable to columns of constant bending stiffness:

when

$$w_0 = \sum m_j \xi^j \quad (2)$$

then for pin ends

$$a_{10} = 2 \sum m_j S(j, 1) \quad (3a)$$

while for clamped ends

$$a_{10} = 2 \sum m_j C(j, 1) \quad (3b)$$

when

$$\kappa_T = \frac{1}{l^2} \sum m_j \xi^j \quad (4)$$

3.3.2 (Cont'd)

then for pin ends (Reference Eq. (3a) of Paragraph 2.3.3.1.1)

$$w_o = \sum \frac{m_j}{(j+1)(j+2)} (\xi^{j+2} - \xi) \quad (5a)$$

and

$$a_{10} = 2 \sum \frac{m_j}{(j+1)(j+2)} [S(j+2, 1) - S(1, 1)] \quad (6a)$$

while for clamped ends (Reference Eq. (4c) of Paragraph 2.3.3.1.1)

$$w_o = \sum \frac{m_j}{(j+2)(j+2)} (\xi^{j+2} - j \xi^3 - (1-j) \xi^2) \quad (5b)$$

and

$$a_{10} = 2 \sum \frac{m_j}{(j+2)(j+1)} [C(j+2, 1) - j C(3, 1) - (1-j) C(2, 1)] \quad (6b)$$

Values of $S(j, 1)$ and $C(j, 1)$ are found in Tables 2.3.3.1.2-1 and -2 and are summarized in Table 3.3.2-1 below.

TABLE 3.3.2-1. FOURIER COEFFICIENTS FOR POLYNOMIALS

j	S(j, 1)	C(j, 1)
0	.6366	-
1	.3183	0
2	.1893	.05066
3	.1248	.07599
4	.08814	.08592
5	.06541	.08815
6	.05038	.08669

3.3.2 (Cont'd)

For a pinned end column of constant EI and constant linear thermal gradient

$$x_T = \frac{-\Delta \alpha T}{h} = \frac{m_0 \xi^0}{l^2} \text{ we obtain from Eqs. (5a) and (6a).}$$

$$w_0 = \frac{m_0}{(0+1)(0+2)} (\xi^{0+2} - \xi) = \frac{m_0}{2} (\xi^2 - \xi)$$

$$\therefore w_0(l/2) = \frac{m_0}{2} (.5^2 - .5) = -\frac{m_0}{8} = \frac{l^2 \Delta \alpha T}{8h}$$

$$\text{and } a_{10} = 2 \frac{m_0}{(1)(2)} [S(2, 1) - S(1, 1)]$$

$$\therefore a_{10} = 2 \frac{m_0}{2} [.1893 - .3183] = -.129 m_0 = \frac{.129 l^2 \Delta \alpha T}{h} \quad (7)$$

A second method of obtaining the initial eccentricity is by matching the displacements at a discrete number of points. This requires the solution of the set of simultaneous equations $w_0(\xi) = \sum a_{10} w_1(\xi)$ for the value of a_{10} and is described in Paragraph

2.3.3.2. The approximation $a_{10} \approx w_0(1/2)$ is a solution where only one point, the deflection at the center of the column, was matched. The accuracy would increase with the number of displacements which are matched, although the matching of the center deflection of a pin ended column with a uniform lateral load or thermal gradient results in a satisfactory determination of the initial eccentricity. The accuracy also depends upon the form of the lateral deflection. For example, a matching of the mid-length deflection for a pin-ended column results in a more accurate amplitude of the fundamental mode caused by a uniform thermal gradient (parabolic) than by a constant eccentricity.

3.4 SPECIAL CASES

There exist special situations for which the exact solution is known or for which engineering approximations have been accepted because of experimental data. The plausibility of the approximate formulation, presented in this section, is reviewed by degenerating it to those situations with which the analyst has had some experience.

3.4.1 Linear Material

A material whose modulus is independent of the stress levels ($\frac{\partial E_S}{\partial \sigma} = 0$, $E = E_T = E_S$) is described as a linear material. The value of $(1 - E_{T0}/E_{S0})$ is identically zero and no reduction of bending stiffness occurs. The eccentricity (a_{10}/r) does not affect the stability and buckling occurs when the average axial strain attains a value of $K \left(\frac{r^2}{l^2} \right)$ (with the load equal to $F_E = \frac{K}{l^2} EI$) provided no fiber is stressed to its ultimate strength.

This corresponds to the classical "Euler Column" whose buckling load and strain are well defined because of the linearity of the material beyond the buckling stress. A column with a large slenderness ratio (l/r) would behave in the manner described. Although the nondimensional stress-strain relationship includes the linear case (letting $\beta = 0$), it is recommended that the actual value of β for the structural material be used even for the case of low buckling stresses. This will permit an approximation of the effects of eccentricities in causing extreme fiber stresses that may be beyond the "proportional limit" even when the average stress is very low. The deviation of the stability of a structure composed of material which is non-linear to a slight degree from that of a linear material will be insignificant for small eccentricities but may become significant for large eccentricities.

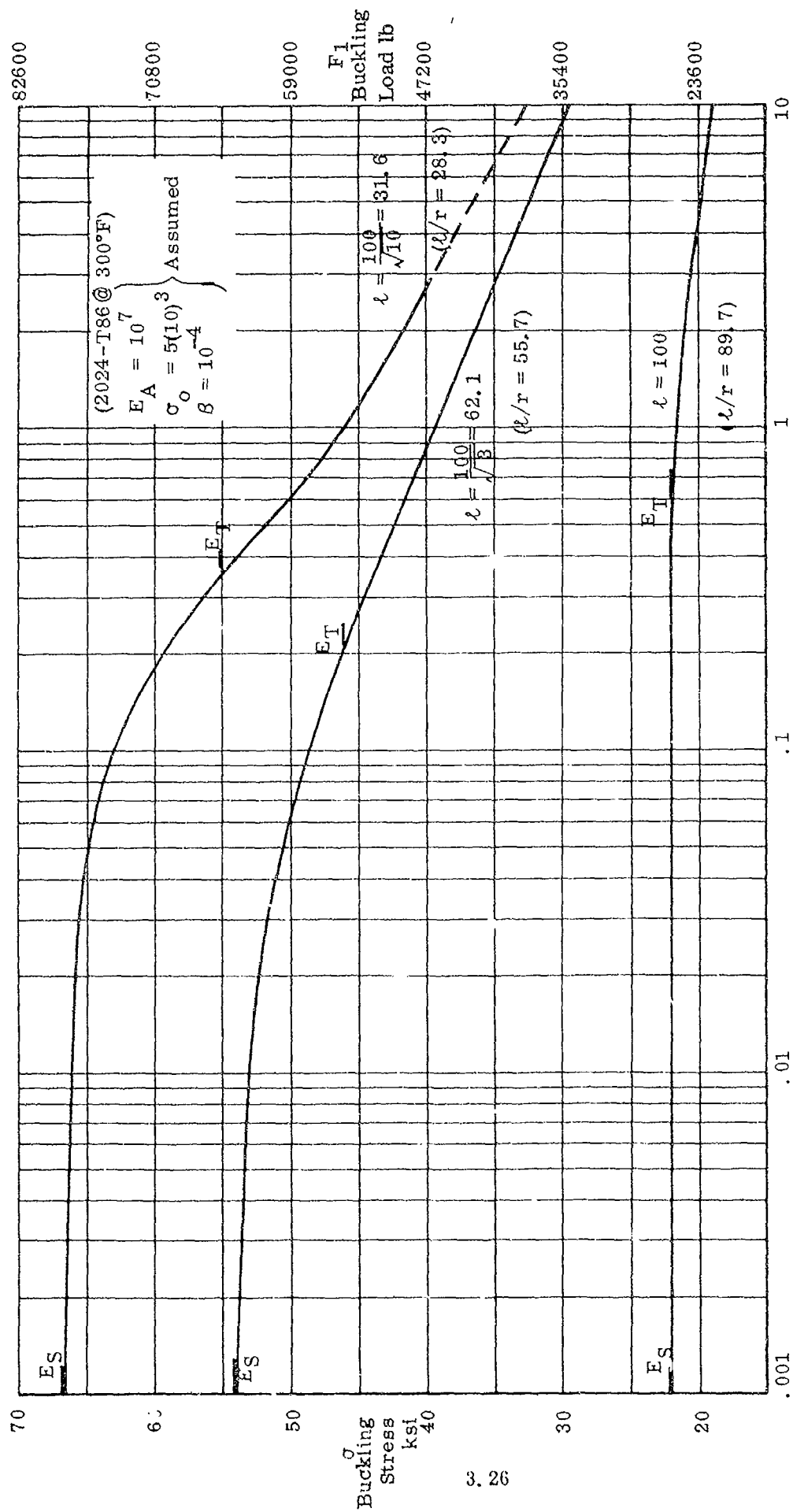
3.4.2 Perfectly Straight Columns

The perfectly straight column does not bend when subjected to an axial load. This condition is virtually impossible to attain experimentally but has some theoretical value as a mathematical model. The column will be stable as long as the axial strain remains below the critical strain $K \frac{r^2}{l^2}$ since a_{10} is zero. Any lateral excitation of the column below this value of the critical strain will dampen out whereas it will magnify and become excessively large when the critical strain is reached or exceeded by the column.

3.4.3 Small Initial Eccentricities

The effect of the eccentricity upon the stability of the column is small when the eccentricity ratio (a_{10}/r) is small. An upper bound upon the stability load will always be F_1 which corresponds to the employment of $E_{S0} I_0$ as the effective buckling stiffness of the cross section and a critical strain of $K \frac{r^2}{l^2}$.

Under the conditions of continuous loadings ($E_{S0} \approx E_{T0}$) and small eccentricities ($1 \gg \left(\frac{\kappa M^2}{\epsilon_0} \right)^2$), the lower bound of the stability load will be $F_1 \left(\frac{E_{T0}}{E_{S0}} \right) = \frac{K}{l^2} E_{T0} I_0$ which



Eccentricity Ratio $\frac{e_{10}}{r}$

FIGURE 3.4.3-1 EFFECT OF ECCENTRICITY ON STABILITY

3.4.3 (Cont'd)

corresponds to the employment of $E_{T_0} I_0$ as the effective buckling stiffness of the cross section and a critical strain of $\left(\frac{E_{T_0}}{E_{S_0}}\right) \left(K \frac{r^2}{l^2}\right)$

This lower bound is satisfactory provided the eccentricity is not too large. The bending stresses become more significant in determining the bending stiffness as the eccentricity increases. An eccentricity ratio exists for each column beyond which the tangent modulus stability ceases to be a lower bound. This eccentricity ratio can be obtained by equating the tangent modulus load to the eccentric column load.

Equality results when
$$\frac{a_{10}}{r} = \sqrt{\frac{1 - E_{T_0}/E_{S_0}}{6}}$$

A value of (a_{10}/r) less than $\sqrt{\frac{1 - E_{T_0}/E_{S_0}}{6}}$ results in a tangent modulus load

that is conservative with respect to the eccentric column load (i.e., $E_{T_0} I_0 \leq \bar{E} I$) whereas a larger eccentricity ratio will make the tangent modulus load unconservative. The value of the critical eccentricity ratio depends upon the material and the slenderness ratio of the column. This is illustrated in Figure 3.4.3-1 which indicates the effect of eccentricity on the stability of the columns described in the illustrative problems. The tangent modulus loads stops being conservative for eccentricity ratios between .2 and .5 for the columns analyzed, and the critical eccentricity ratios are indicated in Figure 3.4.3-1.

3.5 ILLUSTRATIVE PROBLEMS

The computational techniques are illustrated in the following problems:

Find the buckling loads of the following pin ended columns of constant cross section illustrated in Figure 3.5-1.

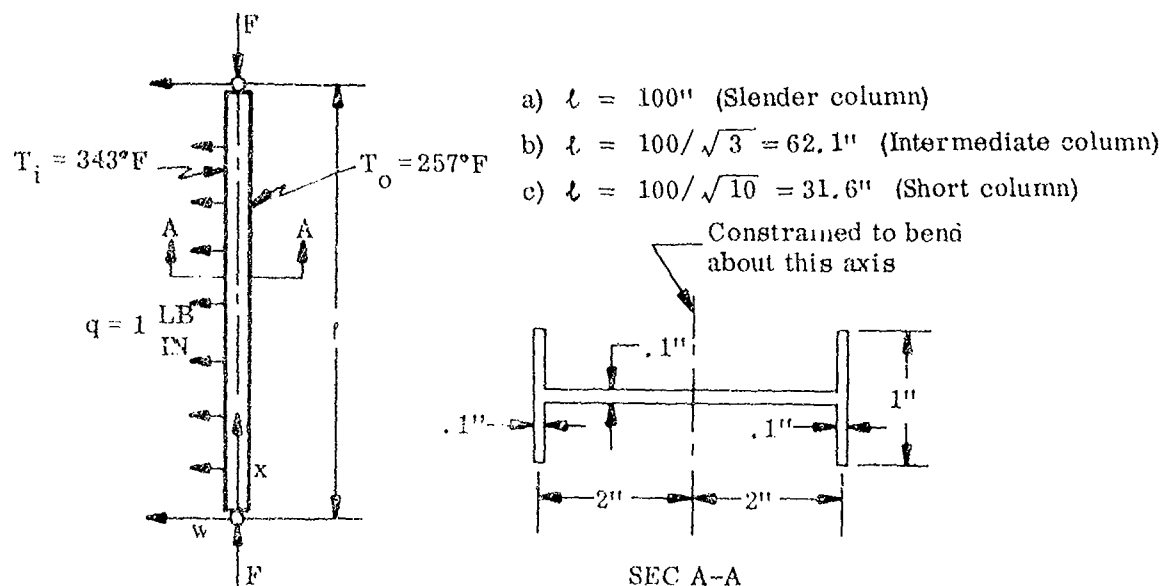


FIGURE 3.5-1 COLUMN SUBJECTED TO LOAD AND TEMPERATURE

3.5 (Cont'd)

The temperature is assumed constant on each face of the column with a linear gradient through the thickness. Material parameters are assumed for 2024-T86 at the mean temperature of 300°F to illustrate the computational technique. The actual values must be determined from standard material tests.

Material Properties

The following material properties are assumed:

$$\begin{aligned}\alpha &= 12(10)^{-6} \text{ in/in/}^{\circ}\text{F} \\ E_A &= 10^7 \text{ psi} \\ \sigma_o &= 5000 \text{ psi} \\ \beta &= 10^{-4}\end{aligned}$$

Initial Geometry

$$A = 0.1 (1.0 + 3.9 + 1.0) = .59 \text{ in}^2$$

$$I_o = 2(.1)(2)^2 + \frac{1}{12} (.1)(3.9)^3 = 1.30 \text{ in}^4$$

$$r^2 = I_o/A = 2.2 \text{ in}^2$$

$$r = 1.15 \text{ in}$$

$$\Phi = \frac{E_A}{\sigma_o} \frac{KI}{Al^2} = \frac{E_A}{\sigma_o} K \frac{r^2}{l^2} = \frac{10^7}{5(10)^3} \pi^2 \frac{2.2}{l^2} = \frac{4.4}{(l/100)^2}$$

$$\epsilon_1 = K \frac{r^2}{l^2} = \pi^2 \frac{2.2}{l^2} = \frac{22}{l^2}$$

Initial Eccentricity

Because the temperature distribution is linear through the thickness we can utilize Eq. (4) of Paragraph 3.3.2.

$$(a_{10})_T = .129 \frac{\Delta \alpha T}{h} l^2 = .129 \frac{\alpha (T_i - T_o)}{h} l^2 \quad (1)$$

$$\therefore \frac{(a_{10})_T}{r} = \frac{.129 \alpha (T_i - T_o) l^2}{r h} = \frac{.129(12)(10)^{-6} (343-257)}{(1.15)(4.1)} = .28 \left(\frac{l}{100} \right)^2$$

3.5 (Cont'd)

From standard reference texts the lateral deflection due to a uniform load q is

$$(w_o)_q = \frac{q l^4}{24 EI} (\xi - 2\xi^2 + \xi^4)$$

From Eq. (3a) of Paragraph 3.3.2

$$(a_{10})_q = 2 \frac{q l^4}{24 EI} [S(1,1) - 2S(2,1) + S(4,1)]$$

From Table 3.3.2-1

$$(a_{10})_q = 2 \frac{q l^4}{24 EI} [.318 - 2(.1248) + (.08814)] = .013066 \frac{q l^4}{EI} \quad (2)$$

$$\therefore \frac{(a_{10})_q}{r} = \frac{.013066 (1) l^4}{(10)^7 (1.3)(1.15)} = .08 \left(\frac{l}{100} \right)^4$$

It should be noted that

$$(w_o)_q \text{ (at } x = l/2) = \frac{5}{384} \frac{q l^4}{EI} = .01302 \frac{q l^4}{EI} \approx .013066 \frac{q l^4}{EI} = (a_{10})_q$$

and

$$(w_o)_T \text{ (at } x = l/2) = .125 \frac{\Delta \alpha T}{h} l^2 \approx .129 \frac{\Delta \alpha T}{h} l^2 = (a_{10})_T$$

This indicates the magnitude of the errors which can be introduced by approximating the amplitude of the fundamental mode by the deflection at the center of the column are quite small for a pin ended column subjected to uniform lateral load or thermal gradient.

The initial eccentricity ratio is then calculated as follows:

$$\left(\frac{a_{10}}{r} \right) = \frac{(a_{10})_T}{r} + \frac{(a_{10})_q}{r} = .28 \left(\frac{l}{100} \right)^2 + .08 \left(\frac{l}{100} \right)^4$$

For

$$l = 100 \quad ; \quad \frac{a_{10}}{r} = .28 + .08 = .36$$

$$l = 100/\sqrt{3} \quad ; \quad \frac{a_{10}}{r} = .093 + .008 = .101$$

$$l = 100/\sqrt{10} \quad ; \quad \frac{a_{10}}{r} = .028 + .0008 = .0288$$

3.5 (Cont'd)

Stability

Referring to the appropriate graph of Figures 3.3.1 results in the determination of $(\bar{\sigma}_1/\sigma_0)$ for the given values of β , $\bar{\Phi}$, and (a_{10}/r) . The effect of the eccentricity, which is expressed in Eq. (11d) of Paragraph 3.2, can be evaluated by comparing the given value of $\bar{\Phi}$ to the value $\bar{\Phi}$ which corresponds to $(\bar{\sigma}_1/\sigma_0)$ for a straight column (Figure 3.3.1-1

with $a_{10}/r = 0$). This value of $\bar{\Phi} = \frac{E_A}{\sigma_0} \bar{\epsilon}_1$, corresponds to a material and geometry parameter for a straight column which would buckle at the same stress as the eccentric column and is a measure of the average axial strain in the column.

a) For $\ell = 100$, $E_A = 10^7$, $\sigma_0 = 5000$, and $\beta = .0001$ we obtain
 $\bar{\Phi} = 4.4$, $\bar{\epsilon}_1 = .0022$, and $\frac{a_{10}}{r} = .36$.

From Figures 3.3.1-7 and -8

$$\frac{\bar{\sigma}_1}{\sigma_0} = 4.4$$

$$\bar{\sigma}_1 = 4.4(5000) = 22000 \text{ psi}$$

$$\bar{F}_1 = \bar{\sigma}_1 A = 22000 (.59) = 13000 \#$$

From Figure 3.3.1-1

$$\bar{\Phi} = 4.4$$

$$\bar{\Phi}/\bar{\Phi} = 4.4/4.4 = 1 \text{ (no effect of eccentricity because of the large slenderness ratio)}$$

$$\text{and } \bar{\epsilon}_1 = \bar{\epsilon}_1 (\bar{\Phi}/\bar{\Phi}) = .0022$$

b) Similarly for $\ell = 100/\sqrt{3}$

$$\bar{\Phi} = 13.2, \bar{\epsilon}_1 = .0066, \text{ and } \frac{a_{10}}{r} = .101$$

From Figure 3.3.1-6

$$\frac{\bar{\sigma}_1}{\sigma_0} = 9.7$$

$$\bar{\sigma}_1 = 9.7(5000) = 48500 \text{ psi}$$

$$\bar{F}_1 = 48500 (.59) = 28500 \#$$

3.5 (Cont'd)

and from Figure 3.1.1-1

$$\bar{\Phi} = 10.5 \quad \bar{\Phi}/\Phi = \frac{10.5}{13.2} = .80$$

$$\therefore \bar{\epsilon}_1 = .0068(.80) = .0053$$

c) For $l = 100/\sqrt{10}$

$$\Phi = 44, \quad \epsilon_1 = .022, \quad \text{and} \quad \frac{a_{10}}{r} = .0288$$

$$\therefore \bar{\sigma}_1/\sigma_0 = 13.1$$

$$\bar{C}_1 = 10.1(5000) = 65500 \text{ psi}$$

$$\bar{F}_1 = 65500(.59) = 38500 \#$$

$$\bar{\Phi} = 38 \quad \bar{\Phi}/\Phi = 38/44 = .87$$

$$\bar{\epsilon}_1 = .022(.87) = .019$$

The eccentricity ratios a_{10}/r were sufficiently close to available values that there was no need to interpolate between graphs. Plots can be made to show the variation in the stability of a given column with the initial eccentricity and can be utilized if interpolation is required. These plots are shown in Figure 3.4.3-1 for the three (long, intermediate and short) columns analyzed above. The values corresponding to an effective modulus of E_S are obtained from Figure 3.3.1-1 $\left(\frac{a_{10}}{r} = 0\right)$ and the values corresponding to an effective

modulus of E_T are obtained from Figure 9.2.1-2 of Reference 3-1. The plots indicate that $\bar{EI} = E_{S0} I_0$ is always unconservative and $\bar{EI} = E_{T0} I_0$ becomes unconservative in

the vicinity of $.2 \approx \frac{a_{10}}{r} \approx .5$ for the analyzed column is. It should be noted that the reduction in the stability of the long column is least while the reduction in the stability of the intermediate column is greatest with the short column affected to an intermediate degree. This is because the reduction in the stability of the eccentric column depends both upon the eccentricity and the stress levels attained by the column. The long column has the largest eccentricity ratio but has very low axial stresses, the short column has very high axial stresses but very small eccentricities; while the intermediate column has high stresses and moderately high eccentricities.

It is interesting to note the contributions of the various factors in reducing the stability of the column below the "Euler Load" $\left(P_E = \frac{KEI_0}{l^2}\right)$. The factors reducing the stability stiffness can be roughly divided into two parts. The first part represents the reduction in the axial stiffness due to the plasticity caused by average stresses which are above the proportional limit of the material. The second part represents additional reductions in

3.5 (Cont'd)

the stability stiffness because of the eccentricity causing a shift in the neutral axis and a rate of change of the bending stiffness.

Equation (2b) of Paragraph 3.2 can be transformed to the following approximate form in order to evaluate quantitatively the destabilizing effects of eccentricity and plasticity.

$$\frac{\bar{F}_1}{F_E} = \frac{E_{S0} I_0}{EI_0} \left(\frac{\bar{EI}}{E_{S0} I_0} + \frac{\kappa}{E_{S0} I_0} \frac{\partial \bar{EI}}{\partial \kappa} \right) \approx \left(\frac{E_{S0} I_0}{EI_0} \right) \left(\frac{\bar{EI}}{E_{S0} I_0} \right) \left(1 - \frac{\kappa}{E_{S0} I_0} \frac{\partial \bar{EI}}{\partial \kappa} \right)$$

where

\bar{F}_1 is the buckling load of the eccentric column

F_E is the Euler buckling load

$\frac{E_{S0} I_0}{EI_0}$ is the factor representing the reduction in axial stiffness

$\frac{\bar{EI}}{E_{S0} I_0}$ is the effect of the shifting of the neutral axis

$\left(1 - \frac{\kappa}{E_{S0} I_0} \frac{\partial \bar{EI}}{\partial \kappa} \right)$ is the effect due to the change in the bending stiffness

The numerical calculations for the destabilizing effects are summarized in Table 3.5-1.

TABLE 3.5-1 STABILITY RATIOS

DESTABILIZING PHENOMENON	LENGTH OF COLUMN		
	$l=100''$	$l=100/\sqrt{3}$	$l=100/\sqrt{10}$
<u>Reduction in Axial Stiffness</u> (Plasticity Effect) $\frac{E_{So} I_o}{EI_o} = \frac{F_1}{F_E}$	1.00	.73	.30
<u>Further Reductions in Stability Stiffness</u> (Eccentricity Effects) a) Shift of Neutral Axis $\frac{\bar{EI}}{E_{So} I_o} = 1 - 2\alpha^2 r^2$	1.00	.93	.955
b) Rate of Change of Bending Stiffness $1 - \frac{\kappa}{E_{So} I_o} \frac{\partial EI}{\partial \kappa} = 1 - 4\alpha^2 r^2$	1.00	.87	.91
c) Critical Strain Ratio-Total Eccentricity Effect $\frac{\bar{\epsilon}_1}{\epsilon_1} = 1 - 6\alpha^2 r^2 = \frac{\bar{F}_1}{F_1}$	1.00	.80	.865
<u>Stability Stiffness Reduction</u> $\frac{F_1}{F_E} = \frac{\bar{F}_1}{F_1} \frac{F_1}{F_L}$	1.00	.58	.26

3.6 REFERENCES

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- 3-2 Switzky, H., "Approximate Solution for the Buckling of Eccentric Columns", Republic Aviation Report No. ARD-679-6, November 1961.
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- 3-4 Bleich, F., "Buckling Strength of Metal Structures", McGraw-Hill Book Co., Inc., 1952.
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SECTION 4
AXISYMMETRIC LARGE DEFLECTIONS OF
CIRCULAR PLATES SUBJECTED TO THERMAL AND MECHANICAL LOADS

by

M. Newman

M. Forray

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CIRCULAR PLATES SUBJECTED TO THERMAL AND MECHANICAL LOADS

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SECTION 4 - AXISYMMETRIC LARGE DEFLECTIONS OF CIRCULAR PLATES SUBJECTED TO THERMAL AND MECHANICAL LOADS

4.1 SUMMARY

This report is concerned with the nonlinear axisymmetric analysis of circular plates with in-plane edge restraint. Both temperature and mechanical loads are accommodated as an extension of investigations performed for the isothermal mechanical loading problem (References 4-1 through 4-4). An exact mathematical formulation within the framework of the v. Karman large strain-displacement relations (Reference 4-5) is developed. The equilibrium equations and boundary conditions are then derived by utilizing the calculus of variations for arbitrary axisymmetrical temperatures and normal distributed loading. The satisfaction of equilibrium and compatibility equations requires the solution of two simultaneous nonlinear ordinary differential equations subject to the prescribed boundary conditions. Analytical solutions of such equations are apparently not possible and therefore numerical procedures must be employed.

A finite difference procedure utilizing "relaxed iterations," developed by H. Keller and E. Reiss (Reference 4-4), and employed by them for the solution of isothermal problems with apparently unlimited load parameter ranges, is used here for combined thermo-mechanical problems. Numerical results are presented for the special case of a simply supported circular plate with radially immovable boundaries, subject to a uniform pressure and an arbitrary temperature variation through the thickness (no planform variation). These results have been obtained for a large range of temperature and load parameters. However, because of space limitations, only a limited amount of data is presented in this report.

4.2 INTRODUCTION

One of the basic assumptions of the classical linear theory of plates is that bending action does not induce significant midplane stretching. It is further assumed that stresses and deformations produced by loads and restraints in the midplane are superposable on the bending solution. Thus, coupling between the two effects is not accommodated by the classical theory. When the deflections are not small compared to the plate thickness, midplane stretching is no longer insignificant, resulting in a nonlinear interaction between bending and membrane stresses. Therefore, large deflection theory must be employed.

It is the purpose of this report to investigate the axisymmetric large deflection problem for circular plates.

The general formulation presented considers arbitrary axisymmetric temperature and pressure variation, where the von Karman large strain-displacement relations (Reference 4-5) are utilized. These assume infinitesimal strains and finite but small normal deflections ($(\frac{dw}{dr})^2$ is of the order of the strains, but small compared to unity). The remaining assumptions are those of classical plate theory. This formulation is more complete than the conventional linear theory in that the results are valid for deflection magnitudes several times the plate thickness. Moreover, buckling and postbuckling behavior are embodied in the analysis.

4.2 (Cont'd)

Numerical results in nondimensional tabular and graphical form are given for a simply supported circular plate, with full boundary restraint to radial movement, subjected to uniform pressure and arbitrary temperature variation through the thickness.

The following symbols are used throughout this section:

b	Plate radius
h	Plate thickness
r	Radial coordinate
\bar{r}	$\frac{r}{b}$, nondimensional radial coordinate
u	Midplane radial displacement
u_b	Midplane radial displacement at plate edge
u^*	Radial displacement
w	Normal deflection
\bar{w}	$\frac{w}{h}$, nondimensional normal deflection
z	Thickness coordinate
D	$\frac{Eh^3}{12(1-\nu^2)}$, flexural rigidity
E	Young's modulus
K	Relaxation parameter
M_r, M_t	Radial and tangential bending moments, respectively
\bar{M}_r, \bar{M}_t	Nondimensional radial and tangential bending moments, respectively
M_T	$\int_{-h/2}^{h/2} E\alpha T z dz$
\bar{M}_T	Nondimensional form of M_T
N_r, N_t	Radial and tangential membrane forces, respectively
\bar{N}_r, \bar{N}_t	Nondimensional radial and tangential membrane forces, respectively

4.2 (Cont'd)

N_T	$\int_{-h/2}^{h/2} E\alpha T dz$
\bar{N}_T	Nondimensional form of N_T
q	Normal pressure
Q	Nondimensional normal pressure
T	Local temperature with respect to an unstressed and undeflected datum
U_0	Strain energy density
V	Total potential energy
α	Coefficient of linear thermal expansion
β	Slope
ϵ_r, ϵ_t	Radial and tangential strains, respectively
$\epsilon_r^0, \epsilon_t^0$	Midplane radial and tangential strains, respectively
λ	Elastic in-plane edge restraint
ν	Poisson's ratio
ϕ	$\frac{\psi b}{D}$, nondimensional stress function
ϕ_i	Finite difference value of ϕ at the i 'th grid point
ψ	Stress function
σ_r, σ_t	Radial and tangential stresses, respectively
θ	$\frac{b}{h} \sqrt{6(1-\nu^2)} \ \theta$
θ_i	Finite difference value of θ at the i 'th grid point

4.3 BASIC EQUATIONS

4.3.1 Stress-Strain - Displacement Relations

The von Kármán large strain - displacement relations for the axisymmetric strains at any point in the plate are given by

$$\begin{aligned}\epsilon_r &= \frac{du^*}{dr} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2 \\ \epsilon_t &= \frac{u^*}{r}\end{aligned}\tag{1}$$

where $u^*(r, z)$ is the radial displacement of the point and $w(r)$ is the deflection normal to the undeflected midplane (Figure 4.3.1-1).

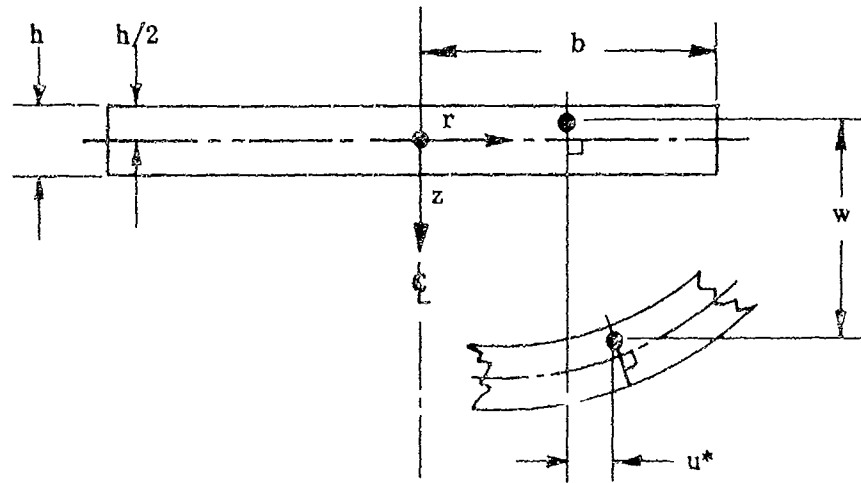


FIGURE 4.3.1-1 GENERAL AXISYMMETRIC DEFLECTION OF A TYPICAL POINT IN THE PLATE

Assuming plane sections to remain plane and normal to the deflected middle surface,

$$u^* = u - z \frac{dw}{dr}\tag{2}$$

4.3.1 (Cont'd)

where $u = u(r)$ is the radial displacement of the midplane. From (1) and (2),

$$\epsilon_r = \frac{du}{dr} - z \frac{d^2 w}{dr^2} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2 \quad (3)$$

$$\epsilon_t = \frac{u}{r} - \frac{z}{r} \frac{dw}{dr}$$

The last (nonlinear) term in the first of (3) does not appear in the conventional small deflection theory. From Hooke's law (neglecting normal stresses in the thickness direction), including the temperature "T", there results

$$\sigma_r = \frac{E}{1-\nu^2} \left[\epsilon_r + \nu \epsilon_t - (1+\nu) \alpha T \right] \quad (4)$$

$$\sigma_t = \frac{E}{1-\nu^2} \left[\epsilon_t + \nu \epsilon_r - (1+\nu) \alpha T \right]$$

An integration of (4) through the plate thickness yields[†]

$$N_r = \int_{-h/2}^{h/2} \sigma_r dz = \frac{Eh}{1-\nu^2} \left[\epsilon_r^0 + \nu \epsilon_t^0 - \frac{(1+\nu)}{Eh} N_T \right] \quad (4a)$$

$$N_t = \int_{-h/2}^{h/2} \sigma_t dz = \frac{Eh}{1-\nu^2} \left[\epsilon_t^0 + \nu \epsilon_r^0 - \frac{(1+\nu)}{Eh} N_T \right]$$

where the midplane strains are given by

$$\epsilon_r^0 = \frac{du}{dr} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2 \quad (4b)$$

$$\epsilon_t^0 = \frac{u}{r} ,$$

[†] In what follows, it is assumed that $\alpha T = \alpha T(r, z)$ but that E (and hence D) is constant.

4.3.1 (Cont'd)

and

$$N_T = \int_{-h/2}^{h/2} E \alpha T dz \quad (4c)$$

Multiplying (4) by z and then integrating through the thickness yields

$$M_r = \int_{-h/2}^{h/2} \sigma_r z dz = -D \left[\frac{d^2 w}{dr^2} + \frac{\nu}{r} \frac{dw}{dr} + \frac{M_T}{D(1-\nu)} \right] \quad (4d)$$

$$M_t = \int_{-h/2}^{h/2} \sigma_t z dz = -D \left[\frac{1}{r} \frac{dw}{dr} + \nu \frac{d^2 w}{dr^2} + \frac{M_T}{D(1-\nu)} \right]$$

where

$$M_T = \int_{-h/2}^{h/2} E \alpha T z dz \quad (4e)$$

and

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

4.3.2 Strain and Potential Energies

The strain energy per unit of volume is given by (Reference 4-6):

$$U_o = \frac{1}{2} \left[\epsilon_r \sigma_r + \epsilon_t \sigma_t - \alpha T (\sigma_r + \sigma_t) \right] \quad (1)$$

Substitution of the stress-strain relations (4) of Paragraph 4.3.1 into (1) yields

$$U_o = \frac{E}{2(1-\nu^2)} \left[\epsilon_r^2 + \epsilon_t^2 + 2\nu \epsilon_r \epsilon_t - 2(1+\nu)(\epsilon_r + \epsilon_t) \alpha T + 2(1+\nu)(\alpha T)^2 \right] \quad (2)$$

For rotationally workless restraints at the outer boundary, the total potential of the plate and external loading system is given by the following equation:

4.3.2 (Cont'd)

$$V = 2\pi \int_0^b r \int_{-h/2}^{h/2} U_0 dz dr - 2\pi \int_0^b q w r dr + \pi b \lambda u_b^2 \quad (3)$$

where q is the normal pressure, λ is an elastic restraint per unit circumferential length to radial displacement of the boundary, and $u_b = u|_{r=b}$. From Eqs. (3) of Paragraph 4.3.1, and (2) and (3) the potential energy in terms of the displacement components and temperature becomes

$$\begin{aligned} V = & \frac{\pi E}{1-\nu} \int_0^b r \left\{ \left[\frac{du}{dr} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2 \right]^2 + \left(\frac{u}{r} \right)^2 + \frac{2\nu u}{r} \left[\frac{dr}{dr} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2 \right] \right\} h \\ & + \left\{ \left(\frac{d^2 w}{dr^2} \right)^2 + \left(\frac{1}{r} \frac{dw}{dr} \right)^2 + \frac{2\nu}{r} \frac{dw}{dr} \frac{d^2 w}{dr^2} \right\} \frac{h^3}{12} \\ & + 2(1+\nu) \left\{ \int_{-h/2}^{h/2} (\alpha T)^2 dz + \frac{M_T}{E} \left[\frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right] \right. \\ & \left. - \frac{N_T}{E} \left[\frac{du}{dr} + \frac{1}{2} \left(\frac{dw}{dr} \right)^2 + \frac{u}{r} \right] \right\} dr - 2\pi \int_0^b q w r dr + \pi b \lambda (u_b)^2 \end{aligned} \quad (4)$$

4.3.3 Governing Differential Equations and Boundary Conditions

The equilibrium equations and "natural" boundary conditions are now obtained by making the potential energy stationary with respect to variations of the displacements w and u ; i.e. $\delta_w V = \delta_u V = 0$. The first of the variations ($\delta_w V = 0$) yields the equilibrium equation

$$D \nabla^4 w - \frac{1}{r} \frac{d}{dr} \left(r N_r \frac{dw}{dr} \right) = q - \frac{\nabla^2 M_T}{1-\nu} \quad (1)$$

and the following boundary conditions:

$$(1) \quad r = 0$$

Assumed regularity of the solution at the center requires that M_r and M_t be finite. This implies that $\frac{dw}{dr} = 0$ and $M_r = M_t$. (2a)

4.3.3 (Cont'd)

(2) $\underline{r = b}$

(i) w prescribed or $-\frac{d}{dr} \left[D \nabla^2 w + \frac{M_T}{1-\nu} \right] + N_r \frac{dw}{dr} = 0$

(2b)

(ii) w' prescribed or $D \left[\frac{d^2 w}{dr^2} + \frac{\nu}{r} \frac{dw}{dr} \right] + \frac{M_T}{1-\nu} = 0$.

The second of the variations ($\delta_u V = 0$) results in

$$N_t - \frac{d}{dr} (r N_r) = 0 \quad (3)$$

with boundary conditions:

(1) $\underline{r = 0}$

N_r and N_t finite. This implies that $u = 0$ (hence $N_r = N_t$). (4a)

(2) $\underline{r = b}$

$\left(u + \frac{N_r}{\lambda} \right) = 0$ or u prescribed. (4b)

The two equations (1) and (3) contain three unknown functions of r ; i.e., w , N_r and N_t . A third equation is obtained from the necessary condition that these three quantities yield a set of single-valued displacements, u and w . A statement of this requirement is obtained by eliminating u from Equation (4b) of Paragraph 4.3.1, which results in

$$\frac{d\epsilon_t^0}{dr} + \frac{\epsilon_t^0 - \epsilon_r^0}{r} = -\frac{1}{2r} \left(\frac{dw}{dr} \right)^2 \quad (5a)$$

Substituting (4a) of Paragraph 4.3.1 into (5a),

$$\frac{d}{dr} \{ N_t - \nu N_r + N_T \} + \frac{(1+\nu)}{r} (N_t - N_r) + \frac{Eh}{2r} \left(\frac{dw}{dr} \right)^2 = 0 \quad (5b)$$

Equation (3) is automatically satisfied by a stress function ψ defined through the relations

$$\psi = r N_r \quad (6)$$

$$\frac{d\psi}{dr} = N_t$$

4.3.3 (Cont'd)

Substitution of (6) into Eqs. (1) and (5b) results in the following coupled set of non-linear differential equations

$$D \nabla^4 w - \frac{1}{r} \frac{d}{dr} \left(\psi \frac{dw}{dr} \right) = q - \frac{1}{1-\nu} \nabla^2 M_T \quad (7)$$

and

$$\frac{d^2 \psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} - \frac{\psi}{r^2} + \frac{Eh}{2r} \left(\frac{dw}{dr} \right)^2 = - \frac{d}{dr} N_T \quad (8)$$

where

$$\nabla^2 = \frac{1}{r} \frac{d}{dr} r \frac{d}{dr}$$

$$\nabla^4 = \nabla^2 \nabla^2$$

A more convenient form is obtained by introducing the slope $\beta = \frac{dw}{dr}$, which yields

$$D \frac{d}{dr} r \frac{d}{dr} \frac{1}{r} \frac{d}{dr} (r\beta) - \frac{d}{dr} (\psi\beta) = qr - \frac{1}{1-\nu} \frac{d}{dr} \left(r \frac{dM_T}{dr} \right) \quad (9)$$

and

$$\frac{d^2 \psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} - \frac{\psi}{r^2} + \frac{Eh}{2r} \beta^2 = - \frac{d}{dr} N_T \quad (10)$$

Equation (9) may now be integrated with respect to r , resulting in

$$D \left(\frac{d^2 \beta}{dr^2} + \frac{1}{r} \frac{d\beta}{dr} - \frac{\beta}{r^2} \right) - \frac{\beta\psi}{r} - \frac{1}{r} \int_0^r q(\eta) \cdot \eta d\eta - \frac{1}{1-\nu} \frac{d}{dr} M_T = 0 \quad (11)$$

The Eqs. (10) and (11) are to be solved subject to the boundary conditions (obtained from (2) and (4))

$$(i) \quad \beta(0) = 0$$

$$(ii) \quad \beta(b) \text{ prescribed (known slope)}$$

(12)

or

$$D \left[\frac{d\beta}{dr} + \frac{\nu}{r} \beta \right]_{r=b} + \left[\frac{M_T}{1-\nu} \right]_{r=b} = 0 \quad (\text{no radial bending moment})$$

$$(iii) \quad \psi(0) = 0 \quad (\text{since } N_r \text{ is finite at } r=0)$$

4.3.3 (Cont'd)

$$(iv) \left[\frac{d\psi}{dr} + \left(\frac{Eh}{b^2\lambda} - \frac{\nu}{b} \right) \psi + N_T \right]_{r=b} = 0 \text{ (elastic radial restraint "\lambda" at } r=b \text{)}$$

or

$$\left[\frac{d\psi}{dr} - \frac{\nu}{r} \psi + N_T \right]_{r=b} \text{ prescribed (known radial displacement at } r=b \text{)}$$

The above differential equations and boundary conditions can be expressed in non-dimensional form as

$$\ddot{\theta} + \frac{\dot{\theta}}{\bar{r}} - \frac{\theta}{\bar{r}^2} - \frac{\theta \phi}{\bar{r}} = \bar{r} Q(\bar{r}) - \bar{M}_T$$

$$\ddot{\phi} + \frac{\dot{\phi}}{\bar{r}} - \frac{\phi}{\bar{r}^2} + \frac{\theta^2}{\bar{r}} = -\bar{N}_T \quad (13)$$

where:

$$\bar{r} = \frac{r}{b}, \quad 0 < \bar{r} < 1$$

$$\left(\cdot = \frac{d}{d\bar{r}} \right)$$

$$\phi = \frac{\psi b}{D}$$

$$\theta = \frac{b}{h} \sqrt{6(1-\nu^2)} \, \theta$$

$$Q(\bar{r}) = \frac{2b^4}{Eh^4} \left[6(1-\nu^2) \right]^{3/2} \frac{1}{\bar{r}^2} \int_0^{\bar{r}} q(\xi) \xi d\xi \quad (14)$$

$$\bar{M}_T = \frac{M_T b^2}{Dh^3} \sqrt{\frac{6(1-\nu)}{1-\nu^2}}$$

$$\bar{N}_T = \frac{N_T b^2}{D}$$

4.3.3 (Cont'd)

The boundary conditions become:

$$(1) \quad \theta(0) = 0$$

$$(2) \quad \phi(0) = 0$$

$$(3) \quad \theta(1) \text{ prescribed}$$

or

$$\theta(1) + \nu \theta(1) + \bar{M}_T(1) = 0 \quad (14a)$$

$$(4) \quad \phi(1) + \left(\frac{Eh}{b\lambda} - \nu \right) \phi(1) + \bar{N}_T(1) = 0$$

or

$$\phi(1) - \nu \phi(1) + \bar{N}_T(1) \text{ prescribed}$$

The set of nonlinear differential Equations (13) and accompanying boundary conditions (14a) are not amenable to analytic solution. Numerical procedures must be used to obtain solutions for specified values of the parameters. A finite-difference approach is presented below together with numerical results.

4.4 NUMERICAL INVESTIGATION

4.4.1 Finite Difference Procedure

The procedure developed in Reference 4-4 is employed here for the combined thermo-mechanical problem. In particular, we consider the case of radially immovable edges ($\lambda \rightarrow \infty$ in (14a) of Paragraph 4.3.3) with simple supports for bending where q , M_T , and N_T are constant (uniform pressure and temperature which varies only through the thickness). For this problem, (13) and (14a) of Paragraph 4.3.3 reduce to

$$\begin{aligned} L\theta &= \theta\phi + Q\bar{r}^2 \\ L\phi &= -\theta^2 \end{aligned} \quad (1)$$

and

$$\begin{aligned} \theta(0) &= 0 \\ \phi(0) &= 0 \\ \dot{\theta}(1) + \nu\theta(1) + \bar{M}_T &= 0 \\ \dot{\phi}(1) - \nu\phi(1) + \bar{N}_T &= 0 \end{aligned} \quad (2)$$

where

$$\begin{aligned} L &= \bar{r} \frac{d}{d\bar{r}} \frac{1}{\bar{r}} \frac{d}{d\bar{r}} \bar{r} \\ Q &= \frac{q}{E} \left(\frac{b}{h} \right)^4 \left[6(1-\nu^2) \right]^{3/2} \end{aligned}$$

and the other nondimensional quantities are as defined previously.

A central difference representation of the differential equations (1) yields the following for the interior points:

$$\begin{aligned} \bar{L}\theta_i &= \theta_i\phi_i + Q\left(\frac{i}{m}\right)^2 \\ \bar{L}\phi_i &= -\theta_i^2 \end{aligned} \quad (i = 1, 2, \dots, m-1) \quad (3)$$

where for an arbitrary function ζ ,

4.4.1 (Cont'd)

$$\bar{L}(\zeta_i) = \frac{im}{i+.5} \left[(i+1) \zeta_{i+1} - i \zeta_i \right] - \frac{im}{i-.5} \left[i \zeta_i - (i-1) \zeta_{i-1} \right] \quad (3a)$$

m = number of subdivisions

$$\bar{r}_i = \frac{i}{m} \quad \begin{array}{c} 0 \quad \frac{1}{m} \quad \frac{2}{m} \quad \cdots \quad \frac{i}{m} \quad \cdots \quad 1 \end{array} \quad \bar{r}$$

The finite difference form for the representation of the boundary conditions may be written as

$$\begin{aligned} \theta_0 &= 0 \\ \zeta_0 &= 0 \\ \theta_m &= \left[\frac{m\theta_{m-1} - \frac{M_T b^2}{Dh} \sqrt{\frac{6(1+v)}{1-v}}}{(m+v)} \right] \\ \phi_m &= \left[\frac{m\phi_{m-1} - \frac{N_T b^2}{D}}{m-v} \right] \end{aligned} \quad (4)$$

where backward differences are used for the end point $\bar{r} = 1$.

Equations (3) and (4) constitute a set of $2(m+1)$ equations in the $2(m+1)$ unknowns θ_i, ζ_i ($i = 0, 1, 2, \dots, m$). However, these algebraic equations are both coupled and nonlinear, and cannot be solved in closed form. A "relaxation iteration" technique is employed, in which the iterative forms of equations (3) are written as

$$\left[\bar{L}\zeta_i \right]_{n+1} = - \left[\theta_i^2 \right]_n \quad (5a)$$

$$\left[\bar{L}\theta_i^* \right]_{n+1} = \left[\theta_i \right]_n \left[\zeta_i \right]_{n+1} + Q \left(\frac{i}{m} \right)^2 \quad (5b)$$

$$\left[\theta_i \right]_{n+1} = K \left[\theta_i^* \right]_{n+1} + [1 - K] \left[\theta_i \right]_n \quad (5c)$$

$$(i = 1, 2, 3, \dots, m-1)$$

where K is a "relaxation" parameter.

A typical iteration, starting with the n 'th set of iterates $\left[\theta_i \right]_n$ is as follows:

- (1) Substitution of $\left[\theta_i \right]_n$ into (5a) and making use of the second and fourth of Eq. (4) yields a setⁿ of $(m-1)$ tri-diagonal, linear algebraic equations from which the quantities $\left[\zeta_i \right]_{n+1}$ are determined.

4.4.1 (Cont'd)

- (2) Substituting $[\theta_i]_n$, $[\phi_i]_{n+1}$ into (5b) again yields a set of linear equations from which the provisional values $[\theta_i^*]_{n+1}$ for the next set of iterates are determined.
- (3) The actual value of the next set of iterates $[\theta_i]_{n+1}$ is obtained from Eq. (5c) where a suitable value of "relaxation" parameter is employed.
- (4) Iterations to convergence are performed.

4.4.2 Numerical Results

Based on the procedure indicated, results are presented in both tabular and graphical form (Table 4.4.2-1 and Figures 4.4.2-1 through 4.4.2-6) for the simply supported solid plate with radially immovable edges.

The nondimensional deflections (\bar{w}), membrane forces (\bar{N}_r and \bar{N}_t), and bending moments (\bar{M}_r and \bar{M}_t), presented in the graphs and tables are defined as follows:

$$\begin{aligned}
 \bar{w} &= \frac{w}{h} \\
 \bar{N}_r &= \frac{N_r b^2}{D} \\
 \bar{N}_t &= \frac{N_t b^2}{D} \\
 \bar{M}_r &= -\frac{M_r b^2}{Dh} \sqrt{6(1-\nu^2)} \\
 \bar{M}_t &= -\frac{M_t b^2}{Dh} \sqrt{6(1-\nu^2)} ;
 \end{aligned} \tag{1}$$

and as defined previously,

$$\begin{aligned}
 \bar{r} &= \frac{r}{b} \\
 \bar{M}_T &= \frac{Eb^2}{Dh} \sqrt{\frac{6(1+\nu)}{1-\nu}} \int_{-h/2}^{h/2} \alpha T z dz \\
 \bar{N}_T &= \frac{Fb^2}{D} \int_{-h/2}^{h/2} \alpha T z dz
 \end{aligned}$$

4.4.2 (Cont'd)

Stresses (in nondimensional form) may be obtained from these formulas (Reference 4-7):

$$\frac{b^2 h \sigma_r}{D} = \frac{12z}{h} \left[\frac{\bar{M}_T - \bar{M}_r}{\sqrt{6(1-\nu^2)}} \right] + \frac{\bar{N}_T}{1-\nu} - 12(1+\nu) \frac{b^2}{h^2} \alpha T + \bar{N}_r \quad (2)$$

$$\frac{b^2 h \sigma_t}{D} = \frac{12z}{h} \left[\frac{\bar{M}_T - \bar{M}_t}{\sqrt{6(1-\nu^2)}} \right] + \frac{\bar{N}_T}{1-\nu} - 12(1+\nu) \frac{b^2}{h^2} \alpha T + \bar{N}_t$$

A discussion of the numerical results follows.

(1) Thermal Buckling

The thermal buckling problem of a circular plate due to an average elevated temperature through the thickness (proportional to \bar{N}_T), where $\bar{M}_T = Q = 0$ and radial edge displacement is prevented, is equivalent to the mechanical buckling problem of a plate having an edge thrust corresponding to a prescribed edge displacement. It is shown in Reference 4-2 that buckling can occur only when the edge thrust exceeds the lowest eigenvalue of the linearized buckling problem. The critical thrust corresponding to this eigenvalue is given by (Reference 4-5):

$$- \left[\sigma_r \right]_{cr} = \frac{(2.05)^2 D}{b^2 h} \quad (3)$$

Since, for the thermal problem, the thrust (up to buckling) is given by

$$\sigma_r = - \frac{N_T}{h(1-\nu)} \quad (4)$$

then from (3), (4), and the definition of \bar{N}_T , we find that

$$\left[\bar{N}_T \right]_{cr} = 2.94$$

The postbuckling behavior of the plate is given by the nonlinear analysis; the numerical results are presented in Figure 4.4.2-1 and the first five sub-tables of Table 4.4.2-1. The table employs a floating decimal number system which is to be interpreted as shown by the following examples:

$$\begin{aligned} 0.5159E 00 &= (0.5159) \times (10^0) = 0.5159 \\ 0.5159E 02 &= (0.5159) \times (10^2) = 51.59 \\ 0.5159E -01 &= (0.5159) \times (10^{-1}) = 0.05159 \end{aligned}$$

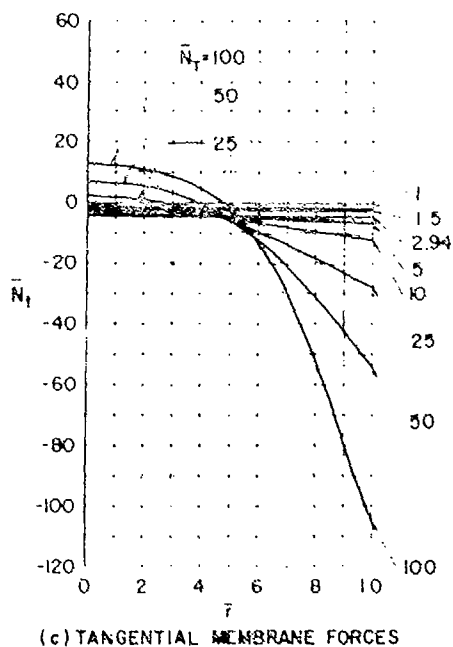
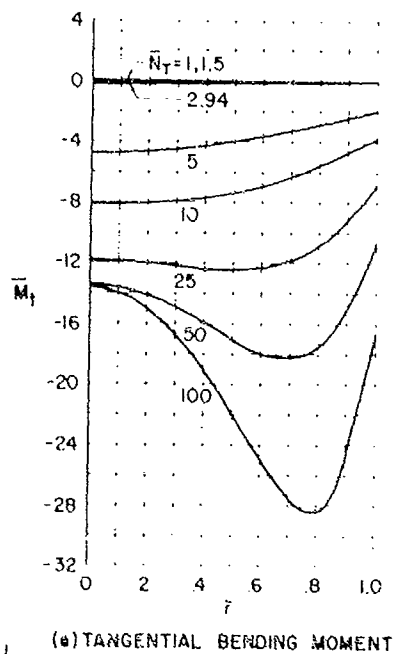
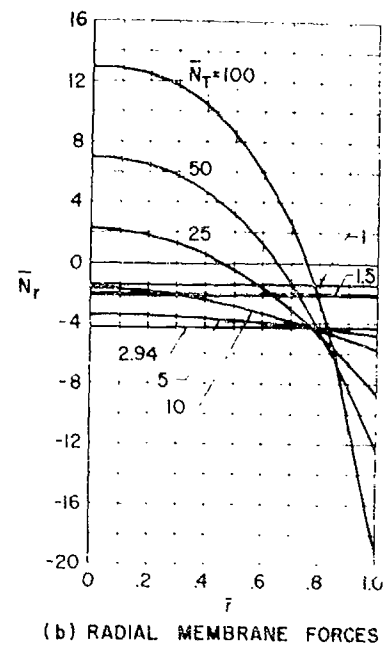
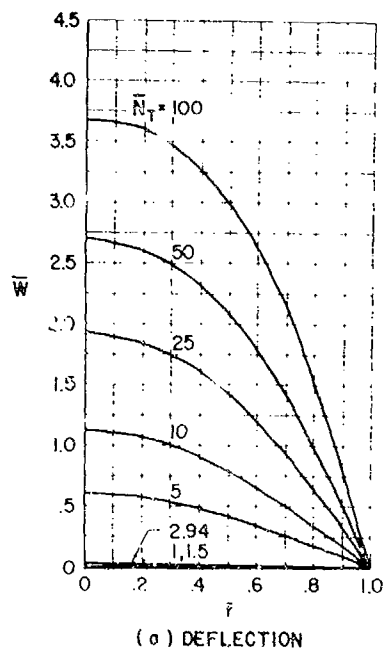


FIGURE 4.4, 2-1 NONDIMENSIONAL THERMAL BUCKLING AND POST BUCKLING BEHAVIOR ($\bar{M}_T = \bar{Q} = 0$).

4.4.2 (Cont'd)

Figure 4.4.2-1 shows the variation of the nondimensional deflection, membrane stress resultants and bending moments with the nondimensional radial coordinate \bar{r} , as \bar{N}_T varies from 0 to 100. It is noted that for $0 < \bar{N}_T < 2.94$ the plate is undeflected, and the two-dimensional linear elastic solution holds, yielding stresses which are constant over the planform and zero bending moments. For higher values of \bar{N}_T , Figure 4.4.2-1a shows nonzero deflections that increase monotonically with increasing \bar{N}_T . For low values of \bar{N}_T , the quantities \bar{N}_r and \bar{N}_t are compressive (Figures 4.4.2-1b and c) throughout the plate. With higher values of \bar{N}_T (25, 50, and 100, for example) deflections in the central region appear to be restrained by tensile membrane stresses, while in the vicinity of the plate edge the membrane stresses become compressive. Figures 4.4.2-1d and e indicate that for small N_T (but above that causing initial buckling), the maximum moments in the radial and tangential directions occur at the center of the plate. With larger \bar{N}_T , maximum moments are away from the center and approach the outer edge as \bar{N}_T increases. The radial moment, in particular, exhibits a strong "boundary layer" effect (sharp gradients near the edge) in meeting the condition $\left[\bar{M}_r \right]_{\bar{r}=1} = 0$.

(2) Bending Due To Combined Thermal And Mechanical Effects

Starting with the sixth sub-table of Table 4.4.2-1, nondimensional numerical results are listed for the thermal bending problem ($Q = 0$), corresponding to the following combinations of thermal parameters:

$$\bar{M}_T = 50, 100, 1000, 2000$$

$$\bar{N}_T = 0, 2.94, 5, 10, 25, 100$$

For a qualitative discussion of the results, we refer to the graphs of Figures 4.4.2-2 through 4.4.2-6 in which mechanical loads ($Q \neq 0$) as well as temperature are considered.

(a) Deflections (Figure 4.4.2-2)

For low pressures (quantities proportional to Q) the deflection in the plate interior becomes constant for large temperature differences (quantities proportional to \bar{M}_T). This flat deflected shape is maintained almost to the plate edges where the boundary requirement of zero deflection causes a boundary layer effect in which the deflections drop precipitously. This effect becomes less pronounced as the pressure loading increases. However, it is interesting to note that even for high pressure loading (Figure 4.4.2-2d), the plate still tends to flatten out in the central region as the temperature difference increases and, contrary to what would be expected, the deflection at the center does not in all cases increase monotonically with the temperature difference.

(b) Membrane Stresses (Figures 4.4.2-3 and 4.4.2-4)

With increasing temperature difference between the plate faces, the radial and tangential stresses tend to become constant and equal to each other in the plate interior,

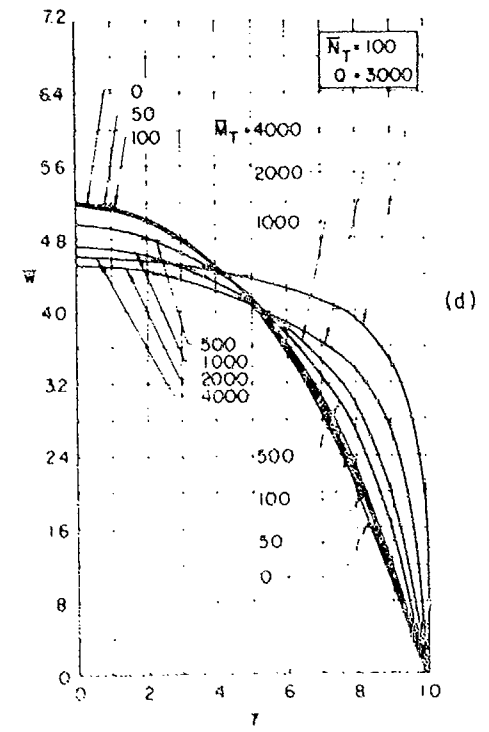
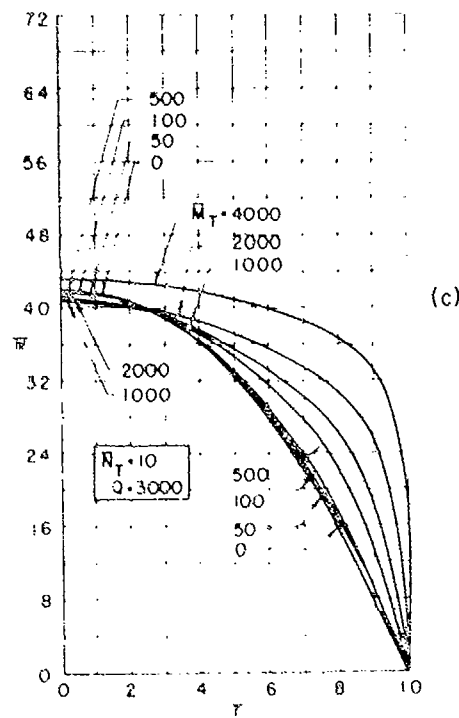
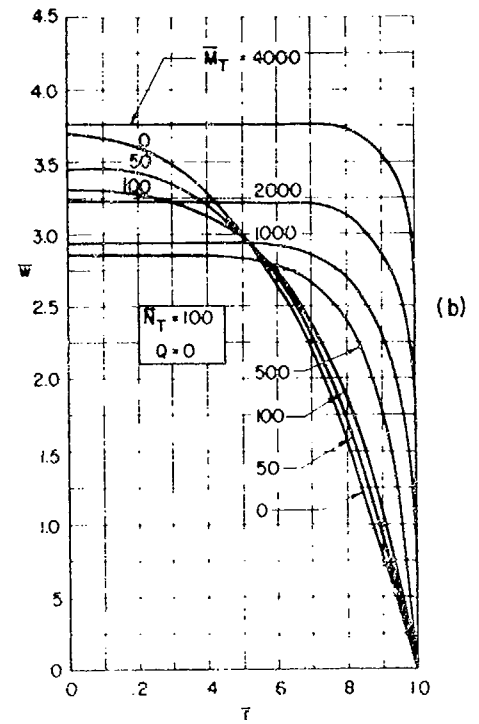
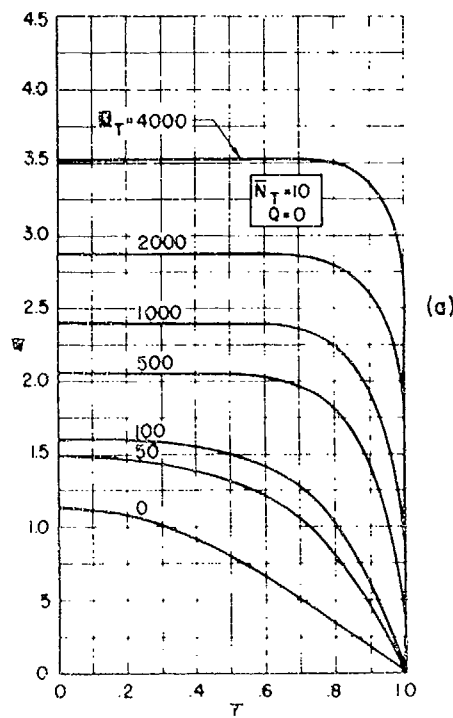


FIGURE 4.4.2-2 NONDIMENSIONAL DEFLECTIONS DUE TO TEMPERATURE WITH AND WITHOUT PRESSURE LOADING

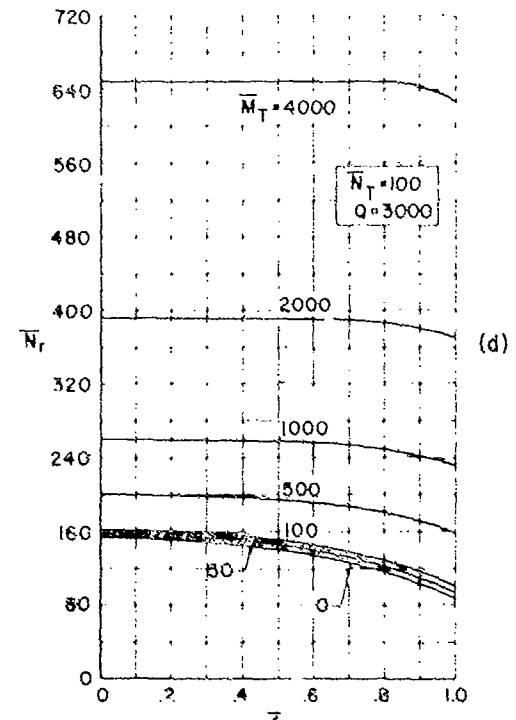
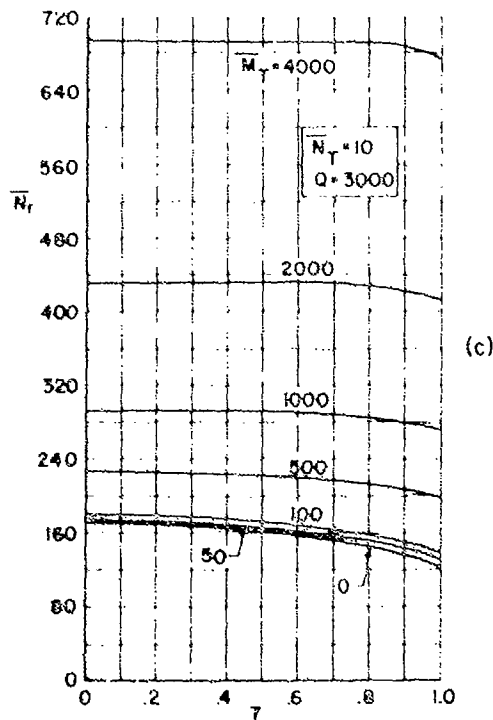
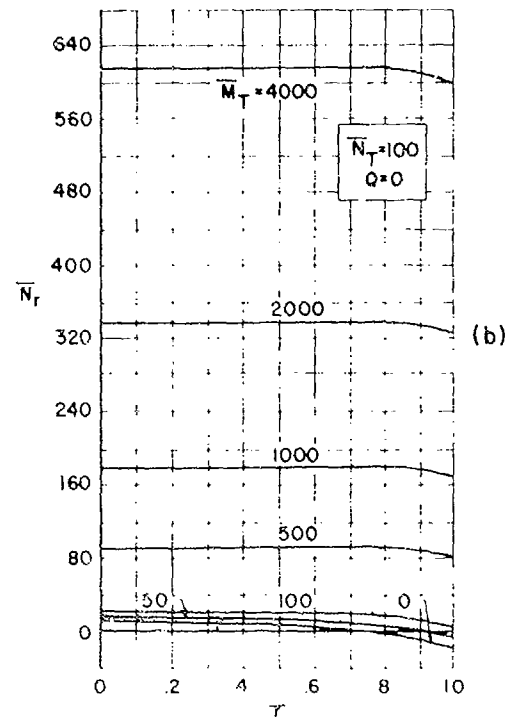
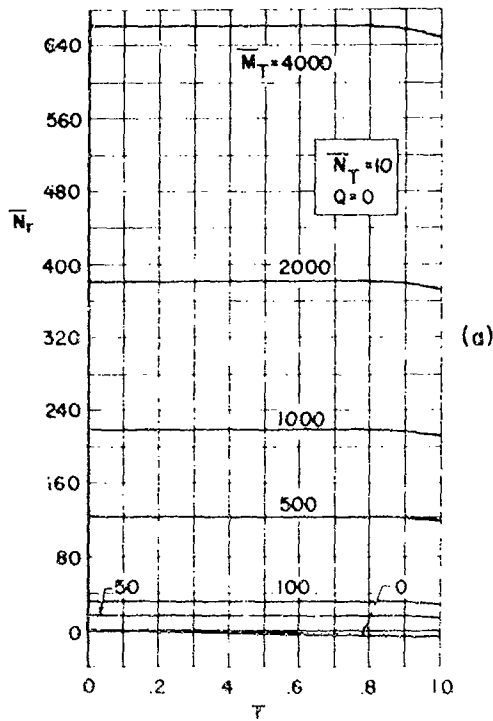


FIGURE 4.4.2-3 NONDIMENSIONAL RADIAL MEMBRANE FORCES DUE TO TEMPERATURE WITH AND WITHOUT PRESSURE LOADING

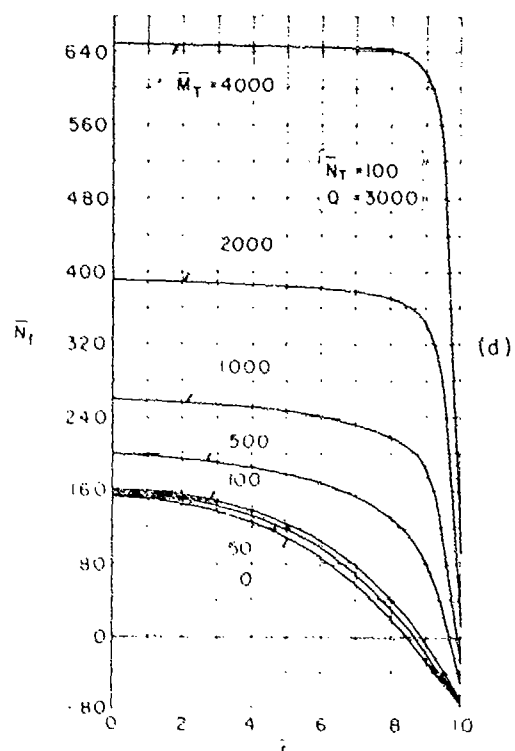
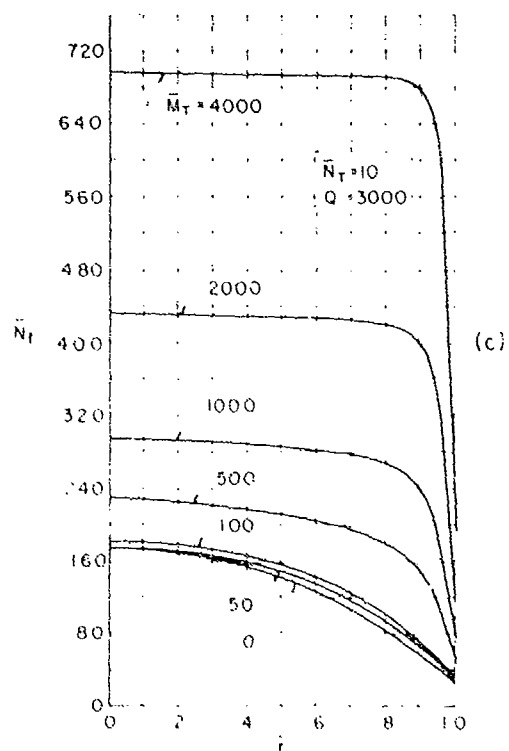
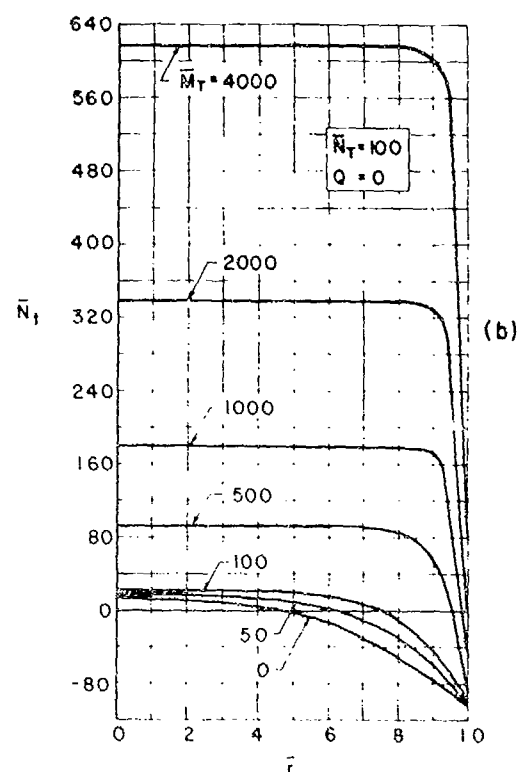
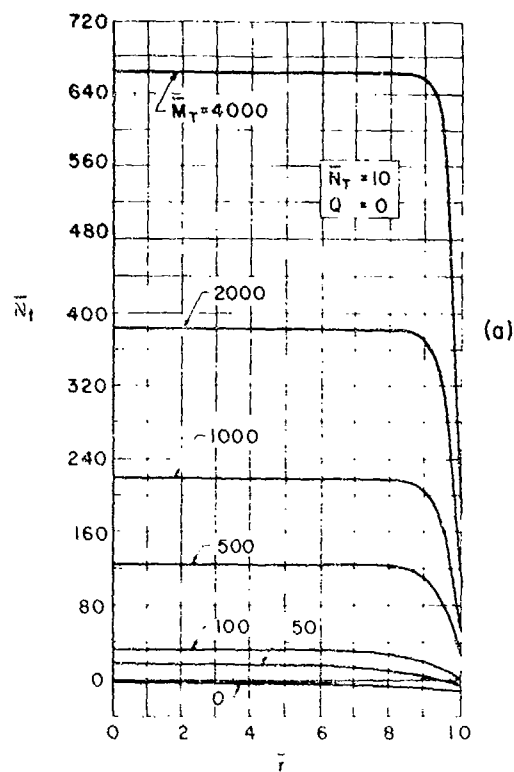


FIGURE 4.4.2-4 NONDIMENSIONAL TANGENTIAL MEMBRANE FORCES DUE TO TEMPERATURE WITH AND WITHOUT PRESSURE LOADING

4.4.2 (Cont'd)

as in a pure membrane. This effect is evident even for high pressure loading, (Figures 4.4.2-3d and 4d). In addition, the magnitudes of these tensile membrane stress resultants increase with increasing temperature difference and normal pressure while they decrease with increasing average temperature (quantities proportional to \bar{N}_T). This is to be expected, since increasing the pressure and temperature difference each cause additional middle plane stretching while \bar{N}_T tends to neutralize this effect. The membrane tension \bar{N}_r , decreases locally as the outer boundary is approached, where the gradients are most pronounced. This negative increment $\Delta\bar{N}_r$, may be responsible for the abrupt reduction in tensile hoop stresses (\bar{N}_t) in the vicinity of the edge (Figure 4.4.2-4). For the larger values of average temperature, the high tensile stress resultants \bar{N}_t , in the plate interior reduce sharply in a boundary layer. Low temperature differences permit transition to compression near the boundary (Figures 4.4.2-4b and d).

(c) Bending Moments (Figures 4.4.2-5 and -6)

The bending moments in the radial and tangential directions are constant and essentially equal over the major central area of the plate. However, radial moments decrease radically near the boundary, satisfying the zero moment boundary condition. Due to the predominantly flat profile of the deflected plate in the interior for large temperature differences (Figures 4.4.2-2a and b), it may be conjectured that the constant and equal interior moments are the same as would occur in a clamped plate subjected to the same temperature gradient (since such a plate will remain flat). To show that this is the case, we proceed as follows:

The bending moments in a fully clamped plate subjected to a thermal gradient through the thickness are given by (Reference 4-8)

$$M_r = M_t = - \frac{M_T}{1-\nu}$$

while

$$w \equiv 0$$

or, in nondimensional form, using the notation of (14) of Paragraph 4.3.3 and (1)

$$\bar{M}_r = \bar{M}_t = \bar{M}_T$$

This result is readily verified by Figures 4.4.2-5 and -6.

ACKNOWLEDGEMENT

The numerical calculations appearing in this section were performed on the Republic Aviation Corporation IBM 7090 digital computer. The authors wish to express their thanks to M. Gershinsky and B. Sackaroff of the Applied Math. Section, Digital Computing Division for their excellent work in coding and supervising the numerical program.

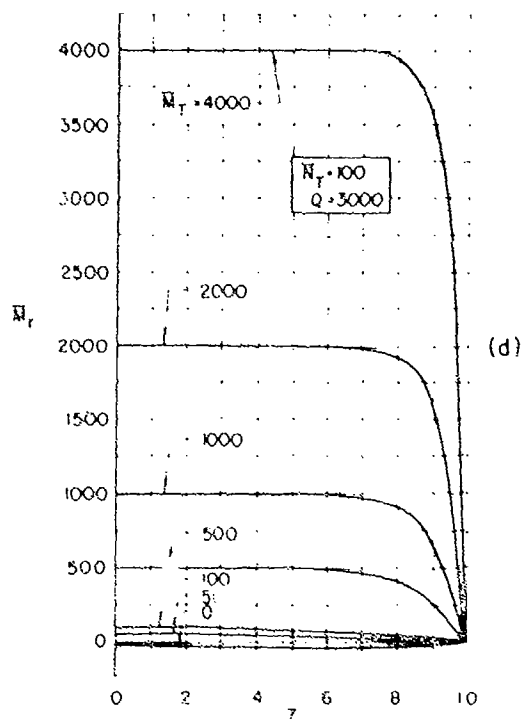
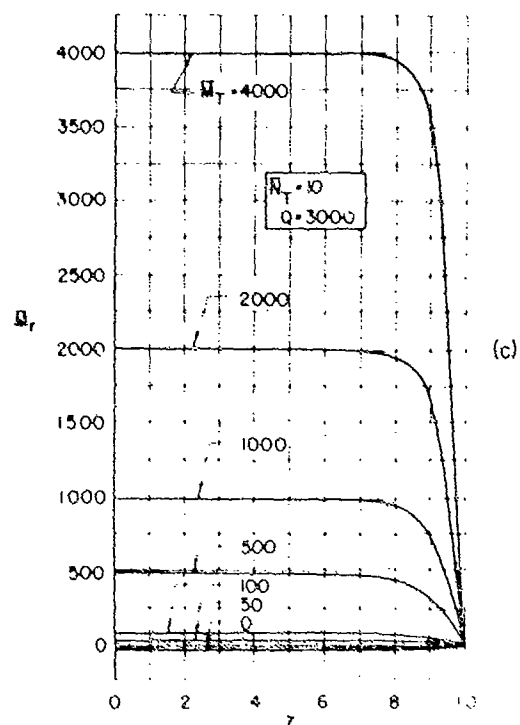
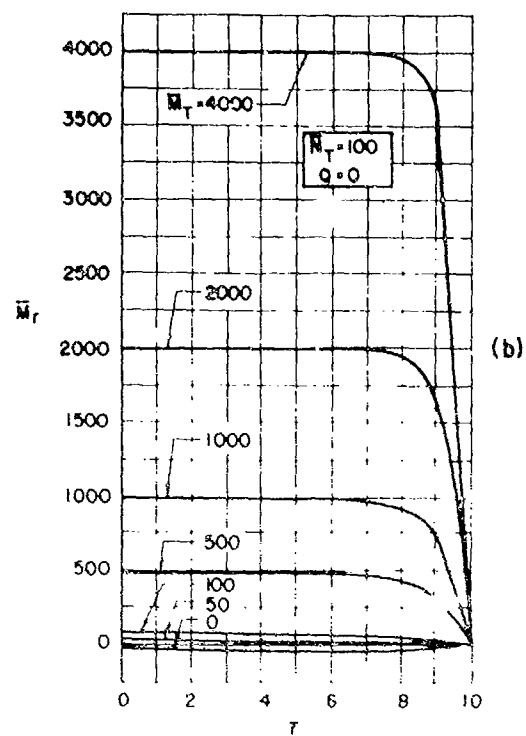
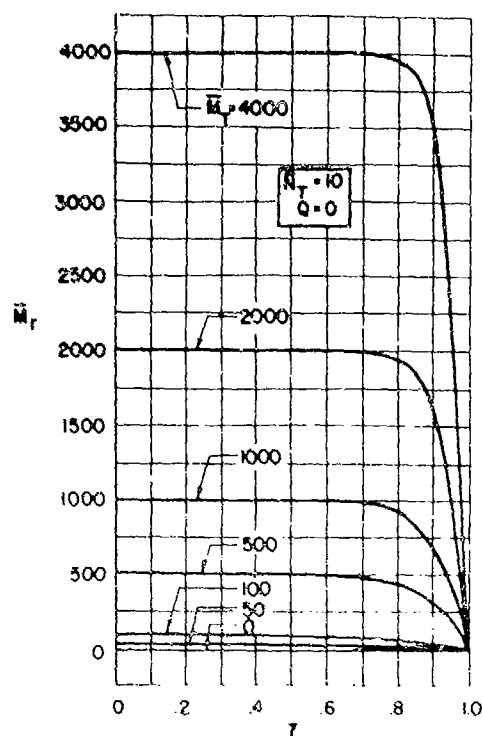


FIGURE 4.4.2-5 NONDIMENSIONAL RADIAL BENDING MOMENTS DUE TO TEMPERATURE WITH AND WITHOUT PRESSURE LOADING

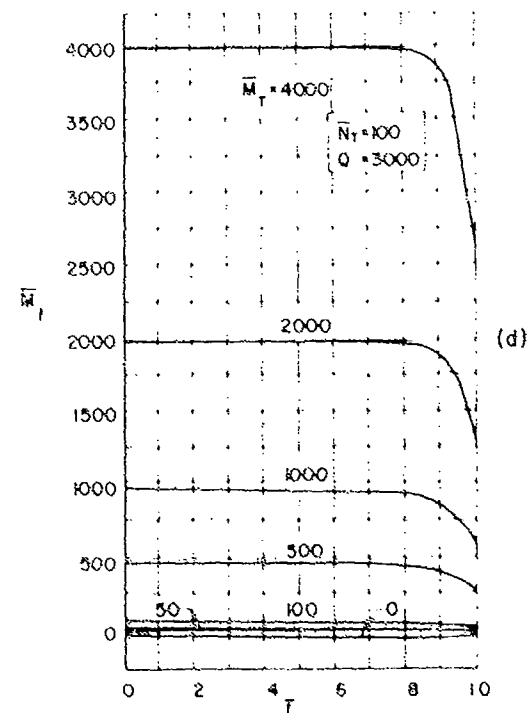
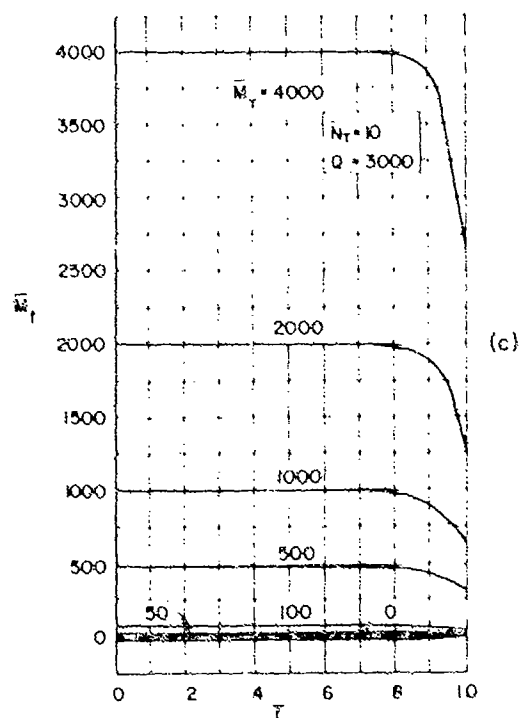
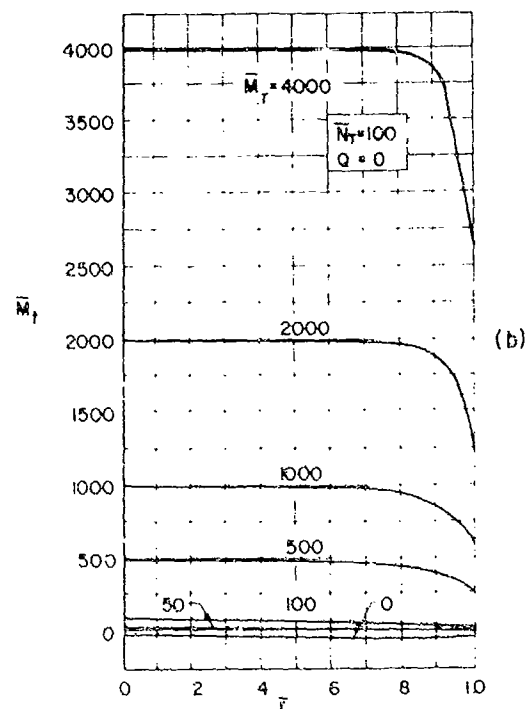
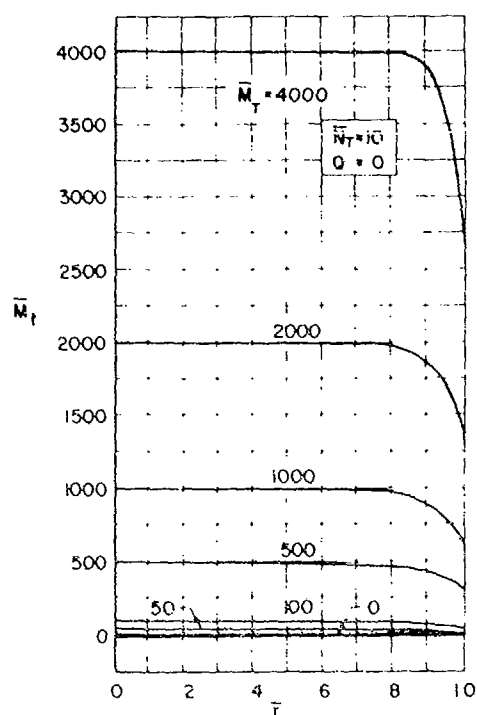


FIGURE 4. 2-6 NONDIMENSIONAL TANGENTIAL BENDING MOMENTS DUE TO TEMPERATURE WITH AND WITHOUT PRESSURE LOADING

4.4.2 (Cont'd)

TABLE 4.4.2-1

NONDIMENSIONAL DEFLECTIONS, MEMBRANE FORCES, AND
BENDING MOMENTS FOR A SIMPLY SUPPORTED CIRCULAR
PLATE WITH RADially IMMOVABLE EDGE

(Pages 4.25 through 4.35)

4.4.2 (Cont'd)

TABLE 4.4.2-1 (Cont'd)

$\bar{N}_T = 0.$				$\bar{N}_T = 0.2940E 01$	
\bar{r}	\bar{w}	\bar{N}_T	\bar{N}_t	\bar{N}_T	\bar{N}_t
0.	0.2600E-01	-0.4193E 01	-0.4193E 01	-0.2072E-00	-0.2072E-00
0.1000E-00	0.2566E-01	-0.4193E 01	-0.4193E 01	-0.0043E-00	-0.2055E-00
0.2000E-00	0.2466E-01	-0.4193E 01	-0.4193E 01	-0.1900E-00	-0.2047E-00
0.3000E-00	0.2300E-01	-0.4193E 01	-0.4193E 01	-0.1824E-00	-0.1985E-00
0.4000E-00	0.2077E-01	-0.4193E 01	-0.4193E 01	-0.1642E-00	-0.1902E-00
0.5000E-00	0.1732E-01	-0.4193E 01	-0.4193E 01	-0.1417E-00	-0.1690E-00
0.6000E-00	0.1286E-01	-0.4193E 01	-0.4193E 01	-0.1160E-00	-0.1476E-00
0.7000E-00	0.1235E-01	-0.4193E 01	-0.4193E 01	-0.0899E-01	-0.1347E-00
0.8000E-00	0.1037E-01	-0.4193E 01	-0.4193E 01	-0.0703E-01	-0.1144E-00
0.9000E-00	0.1014E-01	-0.4193E 01	-0.4193E 01	-0.0730E-01	-0.0985E-01
1.0000E-00	0.	-0.4193E 01	-0.4193E 01	-0.0700E-01	-0.0815E-01
$\bar{N}_T = 0.$				$\bar{N}_T = 0.5000E 01$	
0.	0.2600E-01	-0.4193E 01	-0.4193E 01	-0.4033E 01	-0.4033E 01
0.1000E-00	0.2600E-01	-0.4193E 01	-0.4193E 01	-0.4033E 01	-0.4033E 01
0.2000E-00	0.2600E-01	-0.4193E 01	-0.4193E 01	-0.4033E 01	-0.4033E 01
0.3000E-00	0.2600E-01	-0.4193E 01	-0.4193E 01	-0.4033E 01	-0.4033E 01
0.4000E-00	0.2600E-01	-0.4193E 01	-0.4193E 01	-0.4033E 01	-0.4033E 01
0.5000E-00	0.2600E-01	-0.4193E 01	-0.4193E 01	-0.4033E 01	-0.4033E 01
0.6000E-00	0.2600E-01	-0.4193E 01	-0.4193E 01	-0.4033E 01	-0.4033E 01
0.7000E-00	0.2600E-01	-0.4193E 01	-0.4193E 01	-0.4033E 01	-0.4033E 01
0.8000E-00	0.2600E-01	-0.4193E 01	-0.4193E 01	-0.4033E 01	-0.4033E 01
0.9000E-00	0.2600E-01	-0.4193E 01	-0.4193E 01	-0.4033E 01	-0.4033E 01
1.0000E-00	0.	-0.4193E 01	-0.4193E 01	-0.4033E-00	-0.4033E 01
$\bar{N}_T = 0.$				$\bar{N}_T = 0.7000E 02$	
0.	0.2600E 01	-0.4193E 01	-0.4193E 01	-0.3143E 01	-0.3143E 01
0.1000E-00	0.2600E 01	-0.4193E 01	-0.4193E 01	-0.3143E 01	-0.3143E 01
0.2000E-00	0.2600E 01	-0.4193E 01	-0.4193E 01	-0.3143E 01	-0.3143E 01
0.3000E-00	0.2600E 01	-0.4193E 01	-0.4193E 01	-0.3143E 01	-0.3143E 01
0.4000E-00	0.2600E 01	-0.4193E 01	-0.4193E 01	-0.3143E 01	-0.3143E 01
0.5000E-00	0.2600E 01	-0.4193E 01	-0.4193E 01	-0.3143E 01	-0.3143E 01
0.6000E-00	0.2600E 01	-0.4193E 01	-0.4193E 01	-0.3143E 01	-0.3143E 01
0.7000E-00	0.2600E 01	-0.4193E 01	-0.4193E 01	-0.3143E 01	-0.3143E 01
0.8000E-00	0.2600E 01	-0.4193E 01	-0.4193E 01	-0.3143E 01	-0.3143E 01
0.9000E-00	0.2600E 01	-0.4193E 01	-0.4193E 01	-0.3143E 01	-0.3143E 01
1.0000E-00	0.	-0.4193E 01	-0.4193E 01	-0.3143E-00	-0.3143E 01

4.4.2 (Cont'd)

TABLE 4.4.2-1 (Cont'd)

$\bar{I}_T = 0.$						$\bar{N}_T = 0.2500E 02$					
\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{M}_r	\bar{M}_t	\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{M}_r	\bar{M}_t
0.	0.1936E 01	0.2361E 01	0.2361E 01	0.1179E 02	-0.1179E 02	0.	0.1936E 01	0.2361E 01	0.2361E 01	0.1179E 02	-0.1179E 02
0.1000E-00	0.1916E 01	0.2250E 01	0.2039E 01	-0.1188E 02	-0.1184E 02	0.1000E-00	0.1916E 01	0.2250E 01	0.2039E 01	-0.1188E 02	-0.1184E 02
0.2000E-00	0.1857E 01	0.1938E 01	0.1698E 01	-0.1211E 02	-0.1198E 02	0.2000E-00	0.1857E 01	0.1938E 01	0.1698E 01	-0.1211E 02	-0.1198E 02
0.3000E-00	0.1759E 01	0.1411E 01	-0.5052E 00	-0.1212E 02	-0.1217E 02	0.3000E-00	0.1759E 01	0.1411E 01	-0.5052E 00	-0.1212E 02	-0.1217E 02
0.4000E-00	0.1619E 01	0.6603E 00	-0.3210E 00	-0.1266E 02	-0.1236E 02	0.4000E-00	0.1619E 01	0.6603E 00	-0.3210E 00	-0.1266E 02	-0.1236E 02
0.5000E-00	0.1437E 01	-0.3210E 00	-0.5852E 01	-0.1269E 02	-0.1245E 02	0.5000E-00	0.1437E 01	-0.3210E 00	-0.5852E 01	-0.1269E 02	-0.1245E 02
0.6000E-00	0.1212E 01	-0.1500E 01	-0.9613E 01	-0.1218E 02	-0.1232E 02	0.6000E-00	0.1212E 01	-0.1500E 01	-0.9613E 01	-0.1218E 02	-0.1232E 02
0.7000E-00	0.9491E 00	-0.3007E 01	-0.1399E 02	-0.1005E 02	-0.1182E 02	0.7000E-00	0.9491E 00	-0.3007E 01	-0.1399E 02	-0.1005E 02	-0.1182E 02
0.8000E-00	0.6514E 00	-0.4677E 01	-0.1878E 02	-0.9402E 01	-0.1000E 02	0.8000E-00	0.6514E 00	-0.4677E 01	-0.1878E 02	-0.9402E 01	-0.1000E 02
0.9000E-00	0.3298E-00	-0.6514E 01	-0.2352E 02	-0.4600E 01	-0.2149E 01	0.9000E-00	0.3298E-00	-0.6514E 01	-0.2352E 02	-0.4600E 01	-0.2149E 01
1.0000E-00	0.	-0.8445E 01	-0.2753E 02	-0.5334E-05	-0.4265E 01	1.0000E-00	0.	-0.8445E 01	-0.2753E 02	-0.5334E-05	-0.4265E 01
$\bar{M}_T = 0.$						$\bar{N}_T = 0.1000E 03$					
0.	0.3684E 01	0.1306E 02	0.1306E 02	-0.1357E 02	-0.1357E 02	0.	0.3684E 01	0.1306E 02	0.1306E 02	-0.1357E 02	-0.1357E 02
0.1000E-00	0.3662E 01	0.1292E 02	0.1263E 02	-0.1414E 02	-0.1392E 02	0.1000E-00	0.3662E 01	0.1292E 02	0.1263E 02	-0.1414E 02	-0.1392E 02
0.2000E-00	0.3592E 01	0.1248E 02	0.1128E 02	-0.1587E 02	-0.1490E 02	0.2000E-00	0.3592E 01	0.1248E 02	0.1128E 02	-0.1587E 02	-0.1490E 02
0.3000E-00	0.3468E 01	0.1170E 02	0.8718E 01	-0.1877E 02	-0.1656E 02	0.3000E-00	0.3468E 01	0.1170E 02	0.8718E 01	-0.1877E 02	-0.1656E 02
0.4000E-00	0.3272E 01	0.1046E 02	0.4373E 01	-0.2281E 02	-0.1889E 02	0.4000E-00	0.3272E 01	0.1046E 02	0.4373E 01	-0.2281E 02	-0.1889E 02
0.5000E-00	0.2998E 01	0.8602E 01	-0.2625E 01	-0.2773E 02	-0.2178E 02	0.5000E-00	0.2998E 01	0.8602E 01	-0.2625E 01	-0.2773E 02	-0.2178E 02
0.6000E-00	0.2639E 01	0.5893E 01	-0.1350E 02	-0.3274E 02	-0.2192E 02	0.6000E-00	0.2639E 01	0.5893E 01	-0.1350E 02	-0.3274E 02	-0.2192E 02
0.7000E-00	0.2151E 01	0.2042E 01	-0.2968E 02	-0.3601E 02	-0.2757E 02	0.7000E-00	0.2151E 01	0.2042E 01	-0.2968E 02	-0.3601E 02	-0.2757E 02
0.8000E-00	0.1534E 01	-0.3253E 01	-0.5210E 02	-0.3417E 02	-0.2833E 02	0.8000E-00	0.1534E 01	-0.3253E 01	-0.5210E 02	-0.3417E 02	-0.2833E 02
0.9000E-00	0.7997E 00	-0.1018E 02	-0.7981E 02	-0.2940E 02	-0.2517E 02	0.9000E-00	0.7997E 00	-0.1018E 02	-0.7981E 02	-0.2940E 02	-0.2517E 02
1.0000E-00	0.	-0.1859E 02	-0.1055E 03	0.2771E-04	-0.1406E 02	1.0000E-00	0.	-0.1859E 02	-0.1055E 03	0.2771E-04	-0.1406E 02
$\bar{M}_T = 0.5000E 02$						$\bar{N}_T = 0.$					
0.	0.1180E 01	0.2166E 02	0.2166E 02	0.4794E 02	0.4794E 02	0.	0.1180E 01	0.2166E 02	0.2166E 02	0.4794E 02	0.4794E 02
0.1000E-00	0.1177E 01	0.2164E 02	0.2164E 02	0.4772E 02	0.4772E 02	0.1000E-00	0.1177E 01	0.2164E 02	0.2164E 02	0.4772E 02	0.4772E 02
0.2000E-00	0.1166E 01	0.2164E 02	0.2161E 02	0.4733E 02	0.4733E 02	0.2000E-00	0.1166E 01	0.2164E 02	0.2161E 02	0.4733E 02	0.4733E 02
0.3000E-00	0.1145E 01	0.2162E 02	0.2164E 02	0.4687E 02	0.4710E 02	0.3000E-00	0.1145E 01	0.2162E 02	0.2164E 02	0.4687E 02	0.4710E 02
0.4000E-00	0.1113E 01	0.2159E 02	0.2161E 02	0.4634E 02	0.4660E 02	0.4000E-00	0.1113E 01	0.2159E 02	0.2161E 02	0.4634E 02	0.4660E 02
0.5000E-00	0.1062E 01	0.2153E 02	0.2115E 02	0.4570E 02	0.4590E 02	0.5000E-00	0.1062E 01	0.2153E 02	0.2115E 02	0.4570E 02	0.4590E 02
0.6000E-00	0.9841E 00	0.2143E 02	0.2064E 02	0.4494E 02	0.4548E 02	0.6000E-00	0.9841E 00	0.2143E 02	0.2064E 02	0.4494E 02	0.4548E 02
0.7000E-00	0.8906E 00	0.2125E 02	0.1963E 02	0.4414E 02	0.4492E 02	0.7000E-00	0.8906E 00	0.2125E 02	0.1963E 02	0.4414E 02	0.4492E 02
0.8000E-00	0.7807E 00	0.2094E 02	0.1797E 02	0.4325E 02	0.4371E 02	0.8000E-00	0.7807E 00	0.2094E 02	0.1797E 02	0.4325E 02	0.4371E 02
0.9000E-00	0.6429E 00	0.2036E 02	0.1333E 02	0.4198E 02	0.4164E 02	0.9000E-00	0.6429E 00	0.2036E 02	0.1333E 02	0.4198E 02	0.4164E 02
1.0000E-00	0.	0.1927E 02	0.5783E 01	0.2148E-01	0.2429E 02	1.0000E-00	0.	0.1927E 02	0.5783E 01	0.2148E-01	0.2429E 02

4.4.2 (Cont'd)

TABLE 4.4.2-1 (Cont'd)

$\bar{M}_T = 0.5000E 02$				$\bar{N}_T = 0.2940E 01$	
\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{N}_r	\bar{N}_t
0.	0.1270E 01	0.2027E 02	0.2027E 02	0.4760E 02	0.4760E 02
0.1000E-00	0.1266E 01	0.2027E 02	0.2026E 02	0.4711E 02	0.4751E 02
0.2000E-00	0.1254E 01	0.2025E 02	0.2021E 02	0.4694E 02	0.4722E 02
0.3000E-00	0.1230E 01	0.2022E 02	0.2012E 02	0.4603E 02	0.4671E 02
0.4000E-00	0.1193E 01	0.2018E 02	0.1995E 02	0.4451E 02	0.4590E 02
0.5000E-00	0.1136E 01	0.2010E 02	0.1962E 02	0.4224E 02	0.4468E 02
0.6000E-00	0.1049E 01	0.1998E 02	0.1899E 02	0.3875E 02	0.4288E 02
0.7000E-00	0.9187E 00	0.1976E 02	0.1778E 02	0.3352E 02	0.4025E 02
0.8000E-00	0.7239E 00	0.1938E 02	0.1532E 02	0.2575E 02	0.3644E 02
0.9000E-00	0.4330E-00	0.1871E 02	0.1060E 02	0.1438E 02	0.3098E 02
1.0000E-00	0.	0.1747E 02	0.2329E 01	-0.7152E-05	0.2385E 02
$\bar{M}_T = 0.5000E 02$				$\bar{N}_T = 0.5000E 01$	
0.	0.1334E 01	0.1943E 02	0.1943E 02	0.4736E 02	0.4736E 02
0.1000E-00	0.1330E 01	0.1942E 02	0.1941E 02	0.4718E 02	0.4726E 02
0.2000E-00	0.1316E 01	0.1940E 02	0.1936E 02	0.4666E 02	0.4696E 02
0.3000E-00	0.1271E 01	0.1927E 02	0.1925E 02	0.4570E 02	0.4612E 02
0.4000E-00	0.1250E 01	0.1932E 02	0.1904E 02	0.4415E 02	0.4557E 02
0.5000E-00	0.1188E 01	0.1925E 02	0.1866E 02	0.4178E 02	0.4431E 02
0.6000E-00	0.1094E 01	0.1908E 02	0.1794E 02	0.3814E 02	0.4246E 02
0.7000E-00	0.9561E 00	0.1883E 02	0.1657E 02	0.3290E 02	0.3978E 02
0.8000E-00	0.7507E 00	0.1840E 02	0.1392E 02	0.2513E 02	0.3594E 02
0.9000E-00	0.4472E-00	0.1766E 02	0.8734E 01	0.1392E 02	0.3052E 02
1.0000E-00	0.	0.1631E 02	-0.6811E-01	-0.4758E-06	0.2355E 02
$\bar{M}_T = 0.5000E 02$				$\bar{N}_T = 0.1000E 02$	
0.	0.1470E 01	0.1776E 02	0.1776E 02	0.4675E 02	0.4675E 02
0.1000E-00	0.1464E 01	0.1775E 02	0.1774E 02	0.4649E 02	0.4663E 02
0.2000E-00	0.1467E 01	0.1773E 02	0.1766E 02	0.4597E 02	0.4630E 02
0.3000E-00	0.1436E 01	0.1768E 02	0.1749E 02	0.4490E 02	0.4570E 02
0.4000E-00	0.1388E 01	0.1760E 02	0.1720E 02	0.4321E 02	0.4477E 02
0.5000E-00	0.1314E 01	0.1747E 02	0.1667E 02	0.4064E 02	0.4340E 02
0.6000E-00	0.1205E 01	0.1727E 02	0.1569E 02	0.3669E 02	0.4143E 02
0.7000E-00	0.1046E 01	0.1693E 02	0.1390E 02	0.2443E 02	0.3864E 02
0.8000E-00	0.8155E 00	0.1636E 02	0.1097E 02	0.2364E 02	0.3474E 02
0.9000E-00	0.4872E-00	0.1541E 02	0.4325E 01	0.1283E 02	0.2941E 02
1.0000E-00	0.	0.1376E 02	-0.5825E 01	-0.8136E-05	0.2282E 02

4.4.2 (Cont'd)

TABLE 4.4.2-1 (Cont'd)

$\bar{N}_T = 0.5000E 02$

$\bar{N}_T = 0.2500E 02$

\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{M}_T	\bar{N}_t
0.	0.1936E 01	0.1510E 02	0.1510E 02	0.4498E 02	0.4498E 02
0.1000E-00	0.1928E 01	0.1508E 02	0.1504E 02	0.4473E 02	0.4483E 02
0.2000E-00	0.1902E 01	0.1502E 02	0.1486E 02	0.4297E 02	0.4440E 02
0.3000E-00	0.1855E 01	0.1491E 02	0.1449E 02	0.4261E 02	0.4363E 02
0.4000E-00	0.1783E 01	0.1474E 02	0.1384E 02	0.4051E 02	0.4247E 02
0.5000E-00	0.1676E 01	0.1446E 02	0.1273E 02	0.3743E 02	0.4080E 02
0.6000E-00	0.1522E 01	0.1402E 02	0.1082E 02	0.3311E 02	0.3848E 02
0.7000E-00	0.1305E 01	0.1335E 02	0.7521E 01	0.2719E 02	0.3537E 02
0.8000E-00	0.1001E 01	0.1230E 02	0.1813E 01	0.1939E 02	0.3131E 02
0.9000E-00	0.47.7E 00	0.1061E 02	-0.8014E 01	0.2722E 01	0.2625E 02
1.0000E-00	0.	0.0000E 01	-0.2234E 02	-0.1233E-04	0.2071E 02

$\bar{N}_T = 0.5000E 02$

$\bar{N}_T = 0.1000E 03$

0.	0.3474E 01	0.1679E 02	0.1679E 02	0.4032E 02	0.4032E 02
0.1000E-00	0.3474E 01	0.1679E 02	0.1679E 02	0.4032E 02	0.4032E 02
0.2000E-00	0.3474E 01	0.1679E 02	0.1679E 02	0.4032E 02	0.4032E 02
0.3000E-00	0.3474E 01	0.1679E 02	0.1679E 02	0.4032E 02	0.4032E 02
0.4000E-00	0.3474E 01	0.1679E 02	0.1679E 02	0.4032E 02	0.4032E 02
0.5000E-00	0.3474E 01	0.1679E 02	0.1679E 02	0.4032E 02	0.4032E 02
0.6000E-00	0.3474E 01	0.1679E 02	0.1679E 02	0.4032E 02	0.4032E 02
0.7000E-00	0.3474E 01	0.1679E 02	0.1679E 02	0.4032E 02	0.4032E 02
0.8000E-00	0.3474E 01	0.1679E 02	0.1679E 02	0.4032E 02	0.4032E 02
0.9000E-00	0.3474E 01	0.1679E 02	0.1679E 02	0.4032E 02	0.4032E 02
1.0000E-00	0.	0.0000E 01	0.0000E 01	0.0000E 01	0.0000E 01

$\bar{N}_T = 0.1000E 03$

$\bar{N}_T = 0.$

0.	0.1386E 01	0.3713E 02	0.3713E 02	0.9591E 02	0.9591E 02
0.1000E-00	0.1386E 01	0.3713E 02	0.3713E 02	0.9591E 02	0.9591E 02
0.2000E-00	0.1386E 01	0.3713E 02	0.3713E 02	0.9591E 02	0.9591E 02
0.3000E-00	0.1386E 01	0.3713E 02	0.3713E 02	0.9591E 02	0.9591E 02
0.4000E-00	0.1386E 01	0.3713E 02	0.3713E 02	0.9591E 02	0.9591E 02
0.5000E-00	0.1386E 01	0.3713E 02	0.3713E 02	0.9591E 02	0.9591E 02
0.6000E-00	0.1386E 01	0.3713E 02	0.3713E 02	0.9591E 02	0.9591E 02
0.7000E-00	0.1386E 01	0.3713E 02	0.3713E 02	0.9591E 02	0.9591E 02
0.8000E-00	0.1386E 01	0.3713E 02	0.3713E 02	0.9591E 02	0.9591E 02
0.9000E-00	0.1386E 01	0.3713E 02	0.3713E 02	0.9591E 02	0.9591E 02
1.0000E-00	0.	0.0000E 01	0.0000E 01	0.0000E 01	0.0000E 01

4.4.2 (Cont'd)

TABLE 4.2-1 (Cont'd)

$$\bar{M}_T = 0.1000E 03$$

$$\bar{N}_T = 0.2940E 01$$

\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{M}_r	\bar{M}_t
0.	0.11450E 01	0.3557E 02	0.3557E 02	0.9876E 02	0.9876E 02
0.1000E-00	0.11448E 01	0.3557E 02	0.3557E 02	0.9861E 02	0.9867E 02
0.2000E-00	0.11442E 01	0.3556E 02	0.3556E 02	0.9843E 02	0.9840E 02
0.3000E-00	0.11438E 01	0.3556E 02	0.3556E 02	0.9819E 02	0.9809E 02
0.4000E-00	0.11434E 01	0.3556E 02	0.3556E 02	0.9795E 02	0.9790E 02
0.5000E-00	0.11430E 01	0.3556E 02	0.3556E 02	0.9771E 02	0.9769E 02
0.6000E-00	0.11426E 01	0.3556E 02	0.3556E 02	0.9747E 02	0.9746E 02
0.7000E-00	0.11422E 01	0.3556E 02	0.3556E 02	0.9723E 02	0.9723E 02
0.8000E-00	0.11418E 01	0.3556E 02	0.3556E 02	0.9699E 02	0.9699E 02
0.9000E-00	0.11414E 01	0.3556E 02	0.3556E 02	0.9675E 02	0.9675E 02
1.0000E-00	0.	0.3235E 02	0.7190E 01	0.1147E-02	0.5319E 02

$$\bar{M}_T = 0.1000E 03$$

$$\bar{N}_T = 0.5000E 01$$

0.	0.11495E 01	0.3462E 02	0.3462E 02	0.9867E 02	0.9867E 02
0.1000E-00	0.11493E 01	0.3462E 02	0.3461E 02	0.9851E 02	0.9858E 02
0.2000E-00	0.11485E 01	0.3461E 02	0.3460E 02	0.9830E 02	0.9830E 02
0.3000E-00	0.11471E 01	0.3460E 02	0.3456E 02	0.9803E 02	0.9776E 02
0.4000E-00	0.11466E 01	0.3458E 02	0.3448E 02	0.9778E 02	0.9763E 02
0.5000E-00	0.11462E 01	0.3455E 02	0.3429E 02	0.9753E 02	0.9729E 02
0.6000E-00	0.11458E 01	0.3454E 02	0.3422E 02	0.9728E 02	0.9704E 02
0.7000E-00	0.11454E 01	0.3453E 02	0.3422E 02	0.9703E 02	0.9679E 02
0.8000E-00	0.11450E 01	0.3452E 02	0.3422E 02	0.9678E 02	0.9654E 02
0.9000E-00	0.11446E 01	0.3451E 02	0.3422E 02	0.9653E 02	0.9629E 02
1.0000E-00	0.	0.3116E 02	0.4734E 01	0.2555E-03	0.5223E 02

$$\bar{M}_T = 0.1000E 03$$

$$\bar{N}_T = 0.1000E 02$$

0.	0.1601E 01	0.3253E 02	0.3253E 02	0.9817E 02	0.9817E 02
0.1000E-00	0.1600E 01	0.3253E 02	0.3253E 02	0.9802E 02	0.9801E 02
0.2000E-00	0.1593E 01	0.3253E 02	0.3251E 02	0.9769E 02	0.9800E 02
0.3000E-00	0.1576E 01	0.3251E 02	0.3246E 02	0.9660E 02	0.9740E 02
0.4000E-00	0.1567E 01	0.3249E 02	0.3235E 02	0.9609E 02	0.9639E 02
0.5000E-00	0.1547E 01	0.3244E 02	0.3209E 02	0.9514E 02	0.9573E 02
0.6000E-00	0.1511E 01	0.3234E 02	0.3151E 02	0.8597E 02	0.9202E 02
0.7000E-00	0.1275E 01	0.3213E 02	0.3009E 02	0.7678E 02	0.8763E 02
0.8000E-00	0.1042E 01	0.3170E 02	0.2660E 02	0.6141E 02	0.8272E 02
0.9000E-00	0.6523E 00	0.3072E 02	0.1764E 02	0.3598E 02	0.6901E 02
1.0000E-00	0.	0.2851E 02	-0.1004E 01	0.5435E-04	0.5229E 02

4.4.2 (Cont'd)

TABLE 4.4.2-1 (Cont'd)

$$\bar{M}_T = 0.1000E 03$$

$$\bar{M}_T = 0.2500E 02$$

\bar{r}	\bar{w}	\bar{N}_T	\bar{N}_t	\bar{H}_T	\bar{M}_t
0.	0.1934E 01	0.2797E 02	0.2797E 02	0.2756E 02	0.9756E 02
0.1000E-00	0.1930E 01	0.2796E 02	0.2795E 02	0.2733E 02	0.9713E 02
0.2000E-00	0.1917E 01	0.2795E 02	0.2790E 02	0.2661E 02	0.9702E 02
0.3000E-00	0.1892E 01	0.2792E 02	0.2779E 02	0.2524E 02	0.9626E 02
0.4000E-00	0.1850E 01	0.2786E 02	0.2756E 02	0.2282E 02	0.9500E 02
0.5000E-00	0.1780E 01	0.2776E 02	0.2708E 02	0.1902E 02	0.9299E 02
0.6000E-00	0.1669E 01	0.2757E 02	0.2601E 02	0.1277E 02	0.8985E 02
0.7000E-00	0.1489E 01	0.2720E 02	0.2366E 02	0.7273E 02	0.8496E 02
0.8000E-00	0.1200E 01	0.2643E 02	0.1831E 02	0.5620E 02	0.7741E 02
0.9000E-00	0.7377E 00	0.2499E 02	0.6061E 01	0.3213E 02	0.6595E 02
1.0000E-00	0.	0.2189E 02	-0.1798E 02	-0.2956E-04	0.5045E 02

$$\bar{M}_T = 0.1000E 03$$

$$\bar{M}_T = 0.1000E 03$$

0.	0.3313E 01	0.2268E 02	0.2268E 02	0.9410E 02	0.9117E 02
0.1000E-00	0.3303E 01	0.2266E 02	0.2260E 02	0.9365E 02	0.9384E 02
0.2000E-00	0.3272E 01	0.2257E 02	0.2232E 02	0.9227E 02	0.9305E 02
0.3000E-00	0.3211E 01	0.2240E 02	0.2174E 02	0.8971E 02	0.9163E 02
0.4000E-00	0.3119E 01	0.2211E 02	0.2059E 02	0.8552E 02	0.8934E 02
0.5000E-00	0.2974E 01	0.2161E 02	0.1853E 02	0.7903E 02	0.8590E 02
0.6000E-00	0.2741E 01	0.2075E 02	0.1395E 02	0.6732E 02	0.8086E 02
0.7000E-00	0.2393E 01	0.1823E 02	0.5233E 01	0.5536E 02	0.7374E 02
0.8000E-00	0.1771E 01	0.1653E 02	-0.1201E 02	0.3601E 02	0.6418E 02
0.9000E-00	0.1101E 01	0.1172E 02	-0.4537E 02	0.1500E 02	0.5270E 02
1.0000E-00	0.	0.3266E 01	-0.9855E 02	-0.3814E-04	0.4256E 02

$$\bar{M}_T = 0.1000E 04$$

$$\bar{M}_T = 0.$$

0.	0.2344E 01	0.2246E 03	0.2243E 03	1.0000E 03	1.0000E 03
0.1000E-00	0.2344E 01	0.2246E 03	0.2246E 03	1.0000E 03	1.0000E 03
0.2000E-00	0.2344E 01	0.2246E 03	0.2246E 03	0.9999E 03	0.9999E 03
0.3000E-00	0.2344E 01	0.2246E 03	0.2246E 03	0.9999E 03	0.9999E 03
0.4000E-00	0.2344E 01	0.2246E 03	0.2245E 03	0.9997E 03	0.9999E 03
0.5000E-00	0.2343E 01	0.2246E 03	0.2245E 03	0.9990E 03	0.9990E 03
0.6000E-00	0.2337E 01	0.2245E 03	0.2245E 03	0.9967E 03	0.9984E 03
0.7000E-00	0.2313E 01	0.2243E 03	0.2244E 03	0.9943E 03	0.9939E 03
0.8000E-00	0.2241E 01	0.2245E 03	0.2235E 03	0.9872E 03	0.9752E 03
0.9000E-00	0.1760E 01	0.2242E 03	0.2107E 03	0.7727E 03	0.8960E 03
1.0000E-00	0.	0.2170E 03	0.6505E 02	0.5314E 00	0.6289E 03

$\bar{M}_T = 0.1000E 04$				$\bar{N}_T = 0.2940E 01$	
\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{M}_r	\bar{M}_t
0.	0.2362E 01	0.2229E 03	0.2229E 03	1.0000E 03	1.0000E 03
0.1000E-00	0.2362E 01	0.2229E 03	0.2229E 03	0.9999E 03	1.0000E 03
0.2000E-00	0.2362E 01	0.2229E 03	0.2229E 03	0.9999E 03	0.9999E 03
0.3000E-00	0.2361E 01	0.2229E 03	0.2229E 03	0.9999E 03	0.9999E 03
0.4000E-00	0.2361E 01	0.2229E 03	0.2229E 03	0.9997E 03	0.9998E 03
0.5000E-00	0.2360E 01	0.2229E 03	0.2229E 03	0.9990E 03	0.9996E 03
0.6000E-00	0.2354E 01	0.2229E 03	0.2229E 03	0.9961E 03	0.9984E 03
0.7000E-00	0.2329E 01	0.2229E 03	0.2228E 03	0.9840E 03	0.9938E 03
0.8000E-00	0.2226E 01	0.2228E 03	0.2220E 03	0.9735E 03	0.9749E 03
0.9000E-00	0.1797E 01	0.2023E 03	0.2090E 03	0.7245E 03	0.8275E 03
1.0000E-00	0.	0.2153E 03	0.6208E 02	0.1335E-01	0.6285E 03

$\bar{M}_T = 0.1000E 04$				$\bar{N}_T = 0.5000E 01$	
\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{M}_r	\bar{M}_t
0.	0.2373E 01	0.2217E 03	0.2217E 03	1.0000E 03	1.0000E 03
0.1000E-00	0.2373E 01	0.2217E 03	0.2217E 03	0.9999E 03	1.0000E 03
0.2000E-00	0.2373E 01	0.2217E 03	0.2217E 03	0.9999E 03	0.9999E 03
0.3000E-00	0.2373E 01	0.2217E 03	0.2217E 03	0.9999E 03	0.9999E 03
0.4000E-00	0.2373E 01	0.2217E 03	0.2217E 03	0.9997E 03	0.9998E 03
0.5000E-00	0.2371E 01	0.2217E 03	0.2217E 03	0.9990E 03	0.9996E 03
0.6000E-00	0.2365E 01	0.2217E 03	0.2217E 03	0.9960E 03	0.9984E 03
0.7000E-00	0.2340E 01	0.2217E 03	0.2217E 03	0.9838E 03	0.9937E 03
0.8000E-00	0.2236E 01	0.2217E 03	0.2208E 03	0.9330E 03	0.9747E 03
0.9000E-00	0.1804E 01	0.2212E 03	0.2076E 03	0.7206E 03	0.8271E 03
1.0000E-00	0.	0.2141E 03	0.5981E 02	0.3875E-02	0.6284E 03

$\bar{M}_T = 0.1000E 04$				$\bar{N}_T = 0.1000E 02$	
\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{M}_r	\bar{M}_t
0.	0.2401E 01	0.2191E 03	0.2191E 03	1.0000E 03	1.0000E 03
0.1000E-00	0.2401E 01	0.2191E 03	0.2191E 03	0.9999E 03	1.0000E 03
0.2000E-00	0.2401E 01	0.2191E 03	0.2191E 03	0.9999E 03	0.9999E 03
0.3000E-00	0.2401E 01	0.2191E 03	0.2191E 03	0.9999E 03	0.9999E 03
0.4000E-00	0.2401E 01	0.2191E 03	0.2191E 03	0.9997E 03	0.9998E 03
0.5000E-00	0.2399E 01	0.2191E 03	0.2191E 03	0.9990E 03	0.9995E 03
0.6000E-00	0.2393E 01	0.2191E 03	0.2191E 03	0.9959E 03	0.9983E 03
0.7000E-00	0.2367E 01	0.2191E 03	0.2190E 03	0.9834E 03	0.9935E 03
0.8000E-00	0.2260E 01	0.2190E 03	0.2181E 03	0.9318E 03	0.9743E 03
0.9000E-00	0.1820E 01	0.2185E 03	0.2044E 03	0.7182E 03	0.8961E 03
1.0000E-00	0.	0.2112E 03	0.5386E 02	0.1136E-02	0.6280E 03

4.4.2 (Cont'd)

TABLE 4.4.2-1 (Cont'd)

$$\bar{M}_T = 0.1000E 04$$

$$\bar{N}_T = 0.2500E 02$$

\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{M}_r	\bar{M}_t
0.	0.2486E 01	0.2114E 03	0.2114E 03	1.0000E 03	1.0000E 03
0.1000E-00	0.2486E 01	0.2114E 03	0.2114E 03	0.9999E 03	0.9999E 03
0.2000E-00	0.2486E 01	0.2114E 03	0.2114E 03	0.9999E 03	0.9999E 03
0.3000E-00	0.2486E 01	0.2114E 03	0.2114E 03	0.9999E 03	0.9999E 03
0.4000E-00	0.2486E 01	0.2114E 03	0.2114E 03	0.9997E 03	0.9998E 03
0.5000E-00	0.2486E 01	0.2114E 03	0.2114E 03	0.9988E 03	0.9995E 03
0.6000E-00	0.2477E 01	0.2114E 03	0.2113E 03	0.9955E 03	0.9981E 03
0.7000E-00	0.2448E 01	0.2113E 03	0.2113E 03	0.9821E 03	0.9930E 03
0.8000E-00	0.2332E 01	0.2113E 03	0.2102E 03	0.9283E 03	0.9728E 03
0.9000E-00	0.1869E 01	0.2107E 03	0.1952E 03	0.7113E 03	0.8932E 03
1.0000E-00	0.	0.2029E 03	0.3657E 02	0.7047E-03	0.6268E 03

$$\bar{M}_T = 0.1000E 04$$

$$\bar{N}_T = 0.1000E 03$$

0.	0.2925E 01	0.1794E 03	0.1794E 03	0.9999E 03	0.9999E 03
0.1000E-00	0.2925E 01	0.1794E 03	0.1794E 03	0.9999E 03	0.9999E 03
0.2000E-00	0.2925E 01	0.1794E 03	0.1794E 03	0.9999E 03	0.9999E 03
0.3000E-00	0.2925E 01	0.1794E 03	0.1794E 03	0.9999E 03	0.9999E 03
0.4000E-00	0.2924E 01	0.1794E 03	0.1794E 03	0.9998E 03	0.9999E 03
0.5000E-00	0.2920E 01	0.1794E 03	0.1794E 03	0.9990E 03	0.9991E 03
0.6000E-00	0.2907E 01	0.1794E 03	0.1794E 03	0.9979E 03	0.9987E 03
0.7000E-00	0.2861E 01	0.1794E 03	0.1792E 03	0.9950E 03	0.9970E 03
0.8000E-00	0.2698E 01	0.1793E 03	0.1773E 03	0.9710E 03	0.9652E 03
0.9000E-00	0.2111E 01	0.1783E 03	0.1536E 03	0.6734E 03	0.7730E 03
1.0000E-00	0.	0.1670E 03	-0.4829E 02	0.4577E-04	0.6210E 03

$$\bar{M}_T = 0.2000E 04$$

$$\bar{N}_T = 0.$$

0.	0.2831E 01	0.3874E 03	0.3874E 03	0.9999E 03	0.9999E 03
0.1000E-00	0.2831E 01	0.3874E 03	0.3874E 03	0.9999E 03	0.9999E 03
0.2000E-00	0.2831E 01	0.3874E 03	0.3874E 03	0.9999E 03	0.9999E 03
0.3000E-00	0.2831E 01	0.3874E 03	0.3873E 03	0.9999E 03	0.9999E 03
0.4000E-00	0.2831E 01	0.3874E 03	0.3873E 03	0.9999E 03	0.9999E 03
0.5000E-00	0.2831E 01	0.3873E 03	0.3873E 03	0.9999E 03	0.9999E 03
0.6000E-00	0.2830E 01	0.3873E 03	0.3873E 03	0.9999E 03	0.9999E 03
0.7000E-00	0.2821E 01	0.3873E 03	0.3872E 03	0.9999E 03	0.9999E 03
0.8000E-00	0.2768E 01	0.3873E 03	0.3869E 03	0.9999E 03	0.9999E 03
0.9000E-00	0.2400E 01	0.3870E 03	0.3760E 03	0.9999E 03	0.9999E 03
1.0000E-00	0.	0.3768E 03	0.1122E 02	0.1270E 01	0.1270E 01

4.4.2 (Cont'd)

TABLE 4.4.2-1 (Cont'd)

$\bar{M}_T = 0.2000E 04$

$\bar{N}_T = 0.2940E 01$

\bar{r}	\bar{w}	\bar{N}_r	\bar{N}_t	\bar{M}_r	\bar{M}_t
0.	0.2842E 01	0.3856E 03	0.3856E 03	0.2000E 04	0.2000E 04
0.1000E-00	0.2842E 01	0.3856E 03	0.3856E 03	0.2000E 04	0.2000E 04
0.2000E-00	0.2842E 01	0.3856E 03	0.3856E 03	0.2000E 04	0.2000E 04
0.3000E-00	0.2842E 01	0.3856E 03	0.3856E 03	0.2000E 04	0.2000E 04
0.4000E-00	0.2842E 01	0.3856E 03	0.3856E 03	0.2000E 04	0.2000E 04
0.5000E-00	0.2842E 01	0.3856E 03	0.3856E 03	0.1999E 04	0.1999E 04
0.6000E-00	0.2841E 01	0.3856E 03	0.3856E 03	0.1998E 04	0.1999E 04
0.7000E-00	0.2832E 01	0.3856E 03	0.3856E 03	0.1991E 04	0.1996E 04
0.8000E-00	0.2778E 01	0.3856E 03	0.3854E 03	0.1944E 04	0.1980E 04
0.9000E-00	0.2416E 01	0.3853E 03	0.3750E 03	0.1631E 04	0.1870E 04
1.0000E-00	0.	0.3751E 03	0.1101E 03	0.5632E-01	0.1286E 04

$\bar{M}_T = 0.2000E 04$

$\bar{N}_T = 0.5000E 01$

0.	0.2850E 01	0.3845E 03	0.3845E 03	0.2000E 04	0.2000E 04
0.1000E-00	0.2850E 01	0.3845E 03	0.3845E 03	0.2000E 04	0.2000E 04
0.2000E-00	0.2850E 01	0.3845E 03	0.3845E 03	0.2000E 04	0.2000E 04
0.3000E-00	0.2850E 01	0.3845E 03	0.3845E 03	0.2000E 04	0.2000E 04
0.4000E-00	0.2850E 01	0.3845E 03	0.3845E 03	0.2000E 04	0.2000E 04
0.5000E-00	0.2850E 01	0.3845E 03	0.3845E 03	0.1999E 04	0.1999E 04
0.6000E-00	0.2849E 01	0.3845E 03	0.3845E 03	0.1998E 04	0.1999E 04
0.7000E-00	0.2840E 01	0.3845E 03	0.3845E 03	0.1991E 04	0.1996E 04
0.8000E-00	0.2785E 01	0.3845E 03	0.3842E 03	0.1944E 04	0.1980E 04
0.9000E-00	0.2471E 01	0.3842E 03	0.3738E 03	0.1630E 04	0.1870E 04
1.0000E-00	0.	0.3732E 03	0.1078E 03	0.1512E-01	0.1286E 04

$\bar{M}_T = 0.2000E 04$

$\bar{N}_T = 0.1000E 02$

0.	0.2869E 01	0.3817E 03	0.3817E 03	0.2000E 04	0.2000E 04
0.1000E-00	0.2869E 01	0.3817E 03	0.3817E 03	0.2000E 04	0.2000E 04
0.2000E-00	0.2869E 01	0.3817E 03	0.3817E 03	0.2000E 04	0.2000E 04
0.3000E-00	0.2869E 01	0.3817E 03	0.3817E 03	0.2000E 04	0.2000E 04
0.4000E-00	0.2869E 01	0.3817E 03	0.3817E 03	0.2000E 04	0.2000E 04
0.5000E-00	0.2869E 01	0.3817E 03	0.3817E 03	0.1999E 04	0.1999E 04
0.6000E-00	0.2867E 01	0.3817E 03	0.3817E 03	0.1998E 04	0.1999E 04
0.7000E-00	0.2859E 01	0.3817E 03	0.3817E 03	0.1991E 04	0.1996E 04
0.8000E-00	0.2803E 01	0.3817E 03	0.3814E 03	0.1943E 04	0.1979E 04
0.9000E-00	0.2434E 01	0.3814E 03	0.3708E 03	0.1627E 04	0.1869E 04
1.0000E-00	0.	0.3710E 03	0.1025E 03	0.3662E-02	0.1286E 04

$$\bar{N}_T = 0.2000E 04$$

$$\bar{N}_T = 0.2500E 02$$

\bar{N}_1	\bar{N}_2	\bar{N}_T	\bar{N}_t	\bar{N}_T	\bar{N}_t
0.	0.2926E 01	0.3738E 03	0.3738E 03	0.2000E 04	0.2000E 04
0.1000E-00	0.2926E 01	0.3738E 03	0.3738E 03	0.2000E 04	0.2000E 04
0.2000E-00	0.2926E 01	0.3738E 03	0.3738E 03	0.2000E 04	0.2000E 04
0.3000E-00	0.2926E 01	0.3738E 03	0.3738E 03	0.2000E 04	0.2000E 04
0.4000E-00	0.2926E 01	0.3738E 03	0.3738E 03	0.2000E 04	0.2000E 04
0.5000E-00	0.2925E 01	0.3738E 03	0.3738E 03	0.1999E 04	0.1999E 04
0.6000E-00	0.2924E 01	0.3738E 03	0.3738E 03	0.1998E 04	0.1999E 04
0.7000E-00	0.2915E 01	0.3738E 03	0.3738E 03	0.1990E 04	0.1996E 04
0.8000E-00	0.2855E 01	0.3738E 03	0.3734E 03	0.1941E 04	0.1979E 04
0.9000E-00	0.2473E 01	0.3734E 03	0.3620E 03	0.1620E 04	0.1866E 04
1.0000E-00	0.	0.3626E 03	0.8473E 02	0.5188E-03	0.1285E 04

$$\bar{N}_T = 0.2000E 04$$

$$\bar{N}_T = 0.1000E 03$$

0.	0.3215E 01	0.3378E 03	0.3378E 03	0.2000E 04	0.2000E 04
0.1000E-00	0.3215E 01	0.3378E 03	0.3378E 03	0.2000E 04	0.2000E 04
0.2000E-00	0.3215E 01	0.3378E 03	0.3378E 03	0.2000E 04	0.2000E 04
0.3000E-00	0.3215E 01	0.3378E 03	0.3378E 03	0.2000E 04	0.2000E 04
0.4000E-00	0.3215E 01	0.3378E 03	0.3378E 03	0.1999E 04	0.2000E 04
0.5000E-00	0.3215E 01	0.3378E 03	0.3378E 03	0.1999E 04	0.1999E 04
0.6000E-00	0.3213E 01	0.3378E 03	0.3378E 03	0.1997E 04	0.1999E 04
0.7000E-00	0.3199E 01	0.3378E 03	0.3378E 03	0.1988E 04	0.1995E 04
0.8000E-00	0.3122E 01	0.3378E 03	0.3373E 03	0.1929E 04	0.1974E 04
0.9000E-00	0.2669E 01	0.3373E 03	0.3216E 03	0.1587E 04	0.1853E 04
1.0000E-00	0.	0.3242E 03	-0.9233E 00	0.2807E-02	0.1280E 04

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SECTION 5

AXISYMMETRIC STRESSES AND DEFLECTIONS IN RINGS
DUE TO THERMAL AND MECHANICAL LOADS

by

M. Newman

M. Forray

SECTION 5

AXISYMMETRIC STRESSES AND DEFLECTIONS IN SHELLS DUE TO THERMAL AND MECHANICAL LOADS

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SECTION 5 - AXISYMMETRIC STRESSES AND DEFLECTIONS IN SHELLS

DUE TO THERMAL AND MECHANICAL LOADS

5.1 SUMMARY

This section considers the axisymmetric stresses and deflections in heated shells under load. Basic equations, applicable to general shells of revolution, are derived in Sub-section 5.3. These are discussed and applied specifically to conical shells (Sub-section 5.4) and cylindrical shells (Sub-section 5.5) where solutions to the differential equations are given.

5.2 INTRODUCTION

The purpose of this section is to present the theory for the linear elastic analysis of shells of revolution subjected to axisymmetric mechanical loads and temperatures. The formulation used is the linearized version of the development given by E. Reissner (Reference 5-1), extended to include the effects of temperature.

The present work is exact within the framework of linear shell theory and removes the following restrictions of the more elementary presentation in Section 7, Volume I (Reference 5-2):

- (1) The material in Volume I was limited to the cases of truncated conical and cylindrical shells.
- (2) The development in that volume was approximate for the conical shells and,
- (3) The temperature was constant through the thickness, i.e., only meridional variation was permitted.

Since the main objective of this section is to develop the general equations governing the analysis of axisymmetric shells, minimal emphasis is placed upon obtaining specific numerical results. However, problems involving interaction between shells and bulkheads are discussed and, for the purposes of illustration, the general solution for conical and cylindrical shells are derived. Numerical results are given for an unrestrained cylindrical shell subjected to a prescribed temperature variation.

The following symbols are used throughout this section:

h Shell thickness

$$t = \frac{\sqrt{Rh}}{\sqrt{3(1-\nu)^2}}$$

r Distance from a general point on the meridian to the axis of revolution

s Meridional coordinate

5.2 (Cont'd)

u	Displacement in the r direction
w	Axial displacement
z	Axial coordinate
D	Flexural rigidity, $\frac{Eh^3}{12(1-\nu^2)}$
E	Young's modulus
G(s), H(s)	Load - temperature functions defined by Eqs. (1b) of Paragraph 5.3.2 and (1b) of Paragraph 5.4.2, respectively
H	Horizontal force per unit of circumferential length
K	Curvature
L	Length
M	Moment per unit of length
N	Force per unit of length
M _T , N _T	$\int_h E\lambda T\zeta d\zeta$, $\int_h E\lambda T d\zeta$, respectively
P	Surface traction (force per unit of area)
P _H , P _V	Horizontal and vertical components of P, respectively
Q	Shear force per unit of circumferential length perpendicular to shell mid-plane
R	Radius of cylinder
T	Temperature rise above unstressed undeflected datum
V	Vertical force per unit of circumferential length
α	$\sqrt{(r')^2 + (z')^2}$
β	Rotation of meridional element
ϵ	Strain
ζ	Thickness coordinate
θ	Azimuth or hoop angle
λ	Coefficient of linear thermal expansion
ν	Poisson's ratio
ξ	Meridional coordinate
ρ	$\frac{4\sqrt{Ehtan^2\phi}}{D}$
σ	Stress
ϕ	Angle between tangent to meridian and the horizontal
ψ	$\frac{1}{2\rho s^2}$

SUBSCRIPTS

c, 1	Refer to shell edges
c, p	Complementary and particular solution, respectively
i, f	Initial and final respectively
s	In the meridional direction
θ	In the azimuth or hoop direction
ξ	In the meridional direction

5.3 BASIC EQUATIONS

5.3.1 Configuration of a Typical Shell Element

Since this study is concerned with axisymmetric problems for a shell of revolution it is sufficient to consider a typical meridional element before and after deformation due to loads and temperature (Figure 5.3.1-1). Positive directions for loads and displacements are as shown in the figure. The equation of the meridional curve is expressed parametrically by

$$\begin{aligned} r &= r(\xi) \\ z &= z(\xi) \end{aligned} \quad (1a)$$

where ξ is the independent variable which defines the position of a general point on the meridional mid-surface curve. However, the quantity ξ may not have the units of length (for example, ξ may denote an angle). In order to define length along the curve in a general manner, we introduce the quantity $\alpha = \alpha(\xi)$, such that a differential arc length ds is given by

$$ds = \alpha d\xi. \quad (1b)$$

But,

$$\begin{aligned} \cos \varphi &= \frac{dr}{ds} = \frac{dr}{\alpha d\xi} = \frac{r'}{\alpha} \\ \sin \varphi &= \frac{dz}{ds} = \frac{dz}{\alpha d\xi} = \frac{z'}{\alpha} \end{aligned} \quad (1c)$$

Therefore,

$$(r')^2 + (z')^2 = \alpha^2, \quad (1d)$$

which defines α .

We consider that the tangent to the meridional curve before deformation makes an angle φ with a radial line normal to the axis of revolution; the angle after deformation is designated by $(\varphi + \beta)$. Other quantities in the figure are self-explanatory or indicated in the nomenclature, and moment vectors are obtained using the right-hand rule.

5.3.2 Equilibrium Equations

In the following derivation the forces per unit of circumferential length are designated by the components (N_ξ, Q) or the statically equivalent components (V, H) . The relationships between these two sets of forces are given by

$$\begin{aligned} Q &= V \cos \varphi - H \sin \varphi \\ N_\xi &= V \sin \varphi + H \cos \varphi \end{aligned} \quad (1)$$

where, in the linear theory, the initial configuration is employed in writing the force equilibrium equations (terms in β neglected).

A free body diagram showing equilibrium of forces (Figure 5.3.2-1) yields the following equations:

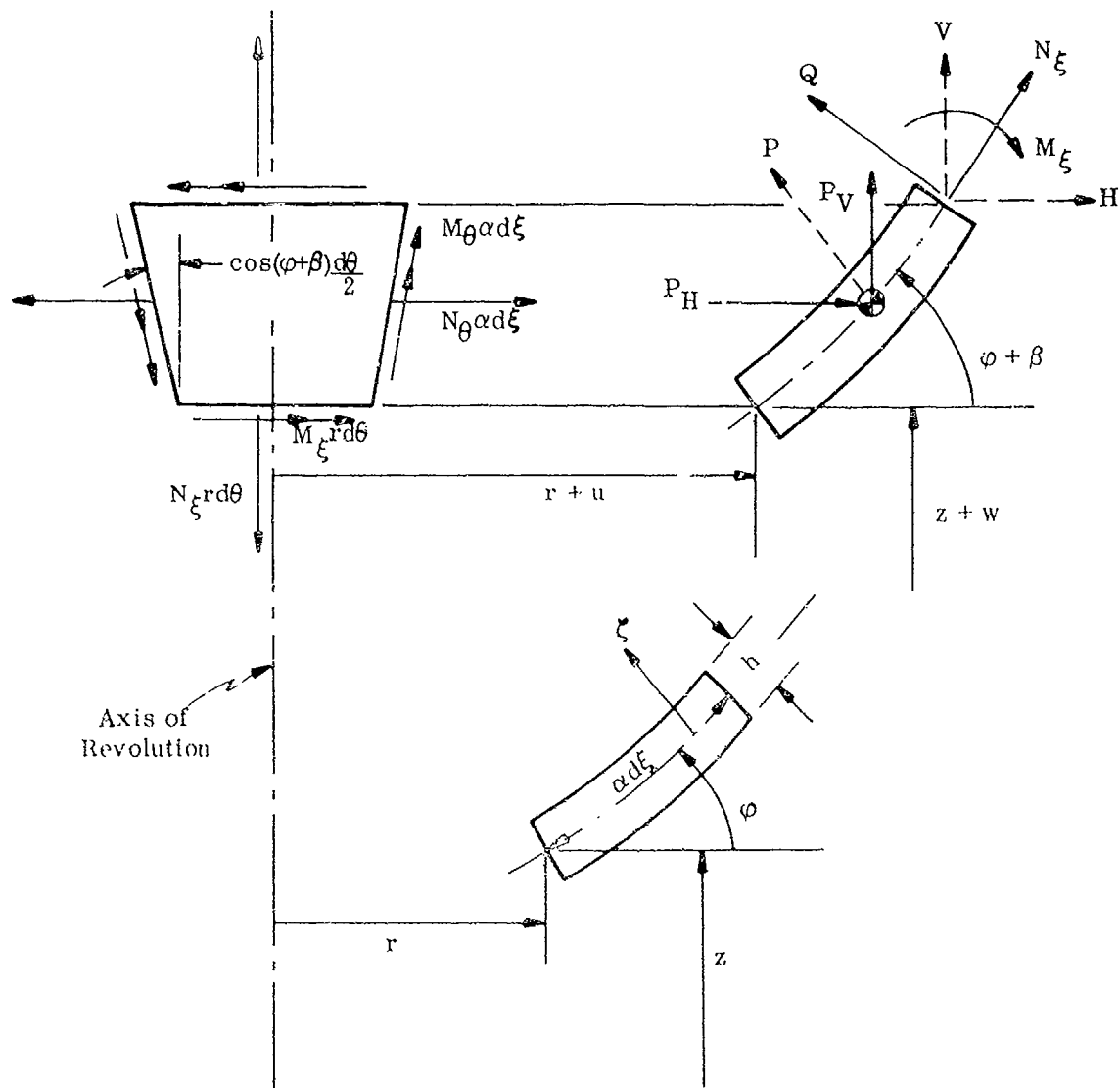


FIGURE 5.3.1-1 CONFIGURATION OF A TYPICAL MERIDIONAL ELEMENT BEFORE AND AFTER DEFORMATION

5.3.2 (Cont'd)

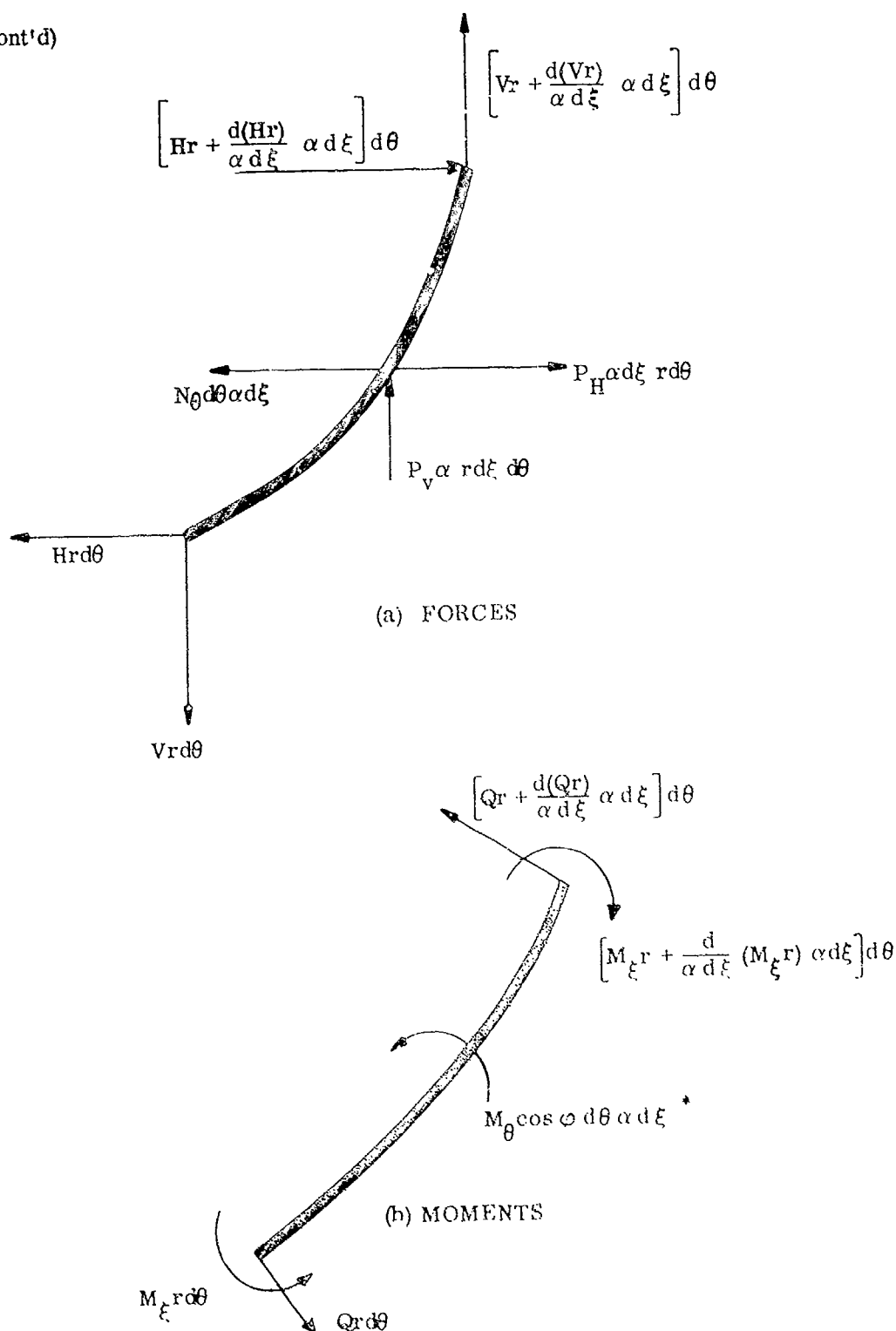


FIGURE 5.3.2-1 EQUILIBRIUM OF FORCES AND MOMENTS

* This represents the vector sum of the hoop moments shown in Figure 5.3.1-1. Since the vectors representing the hoop moments form an angle $\cos \phi d\theta$ (neglecting terms in β), their resultant is of magnitude $M_\theta \cos \phi d\theta \alpha d\xi$ with the direction as indicated in this figure.

5.3.2 (Cont'd)

$$(rV)' + r\alpha P_V = 0 \quad (2a)$$

$$(rH)' - \alpha N_\theta + r\alpha P_H = 0 \quad (2b)$$

From the free body diagram of Figure 5.3.2-1b, moment equilibrium results in

$$(rM_\xi)' - \alpha \cos \varphi M_\theta - \alpha rQ = 0 \quad (3)$$

5.3.3 Curvature Changes and Mid-Surface Strains in Terms of Displacements

Consider a meridional element of the shell mid-surface before deformation (Figure 5.3.1-1). The length of this element is given by

$$dL_i = \frac{dr}{d\xi} d\xi = \frac{r'}{\cos \varphi} d\xi$$

After deformation, the new length is

$$dL_f = \frac{(r' + u') d\xi}{\cos (\varphi + \beta)}$$

where β is the rotation of the element. Thus, the meridional strain of the mid-surface is

$$\begin{aligned} \epsilon_\xi &= \frac{dL_f - dL_i}{dL_i} = \frac{\frac{r' + u'}{\cos (\varphi + \beta)} - \frac{r'}{\cos \varphi}}{\frac{r'}{\cos \varphi}} \\ &= \left(1 + \frac{u'}{r'}\right) \frac{\cos \varphi}{\cos (\varphi + \beta)} - 1 \\ &= \left(1 + \frac{u'}{r'}\right) \cos \varphi \left[\frac{1}{\cos \varphi} + \frac{\beta \sin \varphi}{\cos^2 \varphi} + \dots \right] - 1 \end{aligned}$$

If we neglect terms higher than order β , the linear meridional midsurface strain becomes*

$$\epsilon_\xi = \frac{u'}{r'} + \frac{\sin \varphi}{\cos \varphi} \beta$$

or

$$\epsilon_\xi = \frac{u'}{r'} + \frac{z'}{r'} \beta \quad (1)$$

* It should be noted that the derivations in this section involve division by quantities r and r' . Subsequent derivations also involve division by z' . Therefore, at points where these quantities become zero, the resulting derivations and equations become valid only when appropriate limiting processes and regularity conditions are imposed (see Paragraphs 5.4 and 5.5).

5.3.3 (Cont'd)

The hoop strain of the mid-surface ϵ_θ is a measure of the increase in circumferential length per unit of length and is therefore given by

$$\epsilon_\theta = \frac{u}{r}. \quad (2)$$

In order to obtain bending moments in terms of deformations, it is necessary to determine the changes in principal curvatures of the shell mid-surface. From the definition of curvature, the change in meridional curvature due to deformation is

$$K_\xi = \frac{d(\varphi + \beta)}{\alpha d\xi} = \frac{d\varphi}{\alpha d\xi} = \frac{\beta'}{\alpha} \quad (3a)$$

The hoop principal radius of curvature is defined by the length of the normal line to the surface which is bounded by the surface and the axis of revolution. Thus, the change in hoop curvature due to deformation is

$$K_\theta = \frac{\sin(\varphi + \beta) - \sin \varphi}{r}$$

Again, neglecting terms of order higher than β yields

$$K_\theta = \frac{\beta \cos \varphi}{r} \quad (3b)$$

5.3.4 Stress-Strain and Moment-Curvature Relations

The stress-strain relations for the shell mid-surface including temperature are given by

$$\begin{aligned} \sigma_\xi &= \frac{E}{1-\nu^2} (\epsilon_\xi + \nu \epsilon_\theta) - \frac{E\lambda T}{1-\nu} \\ \sigma_\theta &= \frac{E}{1-\nu^2} (\epsilon_\theta + \nu \epsilon_\xi) - \frac{E\lambda T}{1-\nu} \end{aligned} \quad (1)$$

where λ is the coefficient of linear thermal expansion. Assuming that the strains are linear through the thickness of the shell, the bending strains are odd functions of ζ and therefore integration of the stress-strain relations through the thickness yields

$$\begin{aligned} N_\xi &= \int_{-h}^h \sigma_\xi d\zeta = \frac{Eh}{1-\nu^2} (\epsilon_\xi + \nu \epsilon_\theta) - \frac{N_T}{1-\nu} \\ N_\theta &= \int_{-h}^h \sigma_\theta d\zeta = \frac{Eh}{1-\nu^2} (\epsilon_\theta + \nu \epsilon_\xi) - \frac{N_T}{1-\nu} \end{aligned} \quad (2)$$

where

$$N_T = \int_{-h}^h E\lambda T d\zeta.$$

5.3.4 (Cont'd)

For purposes of obtaining expressions for the moments, it is further assumed that normals to the mid-surface before deformation remain normal after deformation, and that mid-surface stretching due to bending is negligible. Then, employing the stress-strain law, integration of the first moments of the stresses through the thickness results in

$$\begin{aligned} M_{\xi} &= \int_h \sigma_{\xi} \zeta d\zeta = -D \left[K_{\xi} + \nu K_{\theta} \right] - \frac{M_T}{1-\nu} \\ M_{\theta} &= \int_h \sigma_{\theta} \zeta d\zeta = -D \left[K_{\theta} + \nu K_{\xi} \right] - \frac{M_T}{1-\nu} \end{aligned} \quad (3)$$

where

$$\begin{aligned} D &= \frac{Eh^3}{12(1-\nu^2)} \\ M_T &= \int_h E\lambda T \zeta d\zeta . \end{aligned}$$

Substitution of Eqs. (1) - (3) of Paragraph 5.3.3 into Eqs. (2) and (3) gives the force resultants and moments in terms of slopes and displacements:

$$\begin{aligned} N_{\xi} &= \frac{Eh}{1-\nu^2} \left[\frac{u' + z' \beta}{r'} + \frac{\nu u}{r} \right] - \frac{N_T}{1-\nu} \\ N_{\theta} &= \frac{Eh}{1-\nu^2} \left[\frac{u}{r} + \frac{\nu(u' + z' \beta)}{r'} \right] - \frac{N_T}{1-\nu} \end{aligned} \quad (4a)$$

$$\begin{aligned} M_{\xi} &= -D \left[\frac{\beta'}{\alpha} + \frac{\nu \beta r'}{r\alpha} \right] - \frac{M_T}{1-\nu} \\ M_{\theta} &= -D \left[\frac{\beta r'}{r\alpha} + \frac{\nu \beta'}{\alpha} \right] - \frac{M_T}{1-\nu} , \end{aligned} \quad (4b)$$

where $\cos \varphi = \frac{r'}{\alpha}$ has been employed.

A compatibility condition in terms of force resultants and slope may now be obtained. Elimination of the displacement u between Eqs. (1) and (2) of Paragraph 5.3.3 results in

$$\epsilon_{\xi} = \frac{(r_{,\theta})'}{r'} + \frac{z'}{r'} \beta . \quad (5a)$$

The above may be combined with Eqs. (2) to yield

5.3.4 (Cont'd)

$$N_{\xi} - \nu N_{\theta} + N_T = \frac{\left[r (N_{\theta} - \nu N_{\xi} + N_T) \right]'}{r'} + \frac{E h z' \beta}{r'} \quad (5b)$$

5.3.5 Formulation of the Boundary Value Problem

(1) Differential Equation

In the following development, it will be assumed that V is known at one of the shell edges ($V = V_0$ at $\xi = \xi_0$) so that from static equilibrium Eq. (2a) of Paragraph 5.3.2,

$$rV = r_0 V_0 - \int_{\xi_0}^{\xi} r \alpha P_V d\xi. \quad (1)$$

This implies that V is a known quantity throughout the shell.

Then the quantities Q , N_{ξ} , N_{θ} , H , M_{ξ} , M_{θ} may be eliminated from among the seven equations (1), (2a), (2b), (3) of Paragraph 5.3.2, and (4b) and (5b) of Paragraph 5.3.4, to yield a differential equation for the unknown slope β . After detailed calculation and considerable simplification, this equation may be written in the form:

$$\begin{aligned} & \left(\frac{r}{r'} \right)^2 \left[L(\beta) \right]'' + \frac{1}{\alpha r} \left[\left(\frac{\alpha r}{r'} \right)^2 \left(\frac{r}{\alpha} \right) \right]' \left[L(\beta) \right]' + \left[\nu \frac{1}{\alpha} \left(\frac{\alpha r}{r'} \right)' \right. \\ & \left. - 1 - 1 + \frac{1}{r'} \left(\frac{r}{\alpha} \left(\frac{\alpha r}{r'} \right)' \right) \right] L(\beta) + \frac{E h z'}{D r'} \beta - \frac{r^2}{\alpha z'} \frac{V''}{D} + \left[\nu \frac{r}{r'} \left(\frac{z'}{\alpha} \right) \right. \\ & \left. - \frac{z'}{r r'^2} \left(\frac{r^3}{\alpha} \left(\frac{r'}{z'} \right)^2 \right)' \right] \frac{V'}{D} + \left[\left(\frac{r}{r'} \right)^2 \left(\frac{r'^2}{\alpha z'} \right)'' - \frac{1}{\alpha r} \left(\frac{\alpha r}{r'} \right)^2 \left(\frac{r}{\alpha} \right)' \left(\frac{r'^2}{\alpha z'} \right)' \right. \\ & \left. - \left(\frac{r}{\alpha} \right) \left(\frac{\alpha r}{r'} \right)' \left(\frac{r'}{\alpha z'} + \frac{\alpha}{z'} + \nu \frac{1}{r'} \left(\frac{r z'}{\alpha} \right)' - \frac{r r'^2}{\alpha^2 z'} \left(\frac{\alpha}{r'} \right)' \right) \right] \frac{V}{D} \\ & + \frac{1}{D} \left[-\nu r P_H - \frac{1}{r'} \left(r^2 P_H \right)' - \frac{r}{r'} N_T' \right] \left(\frac{r}{r'} \right)^2 \left(\frac{r'}{z' \alpha} \right) \frac{M_T'}{D(1-\nu)} \\ & - \frac{1}{\alpha r} \left[\left(\frac{\alpha r}{r'} \right)^2 \left(\frac{r}{\alpha} \right)' \left(\frac{r'}{z' \alpha} \right) \frac{M_T'}{D(1-\nu)} \right]' - \left[\nu \frac{1}{\alpha} \left(\frac{\alpha r}{r'} \right)' - 1 - 1 \right. \\ & \left. + \frac{1}{r'} \left(\frac{r}{\alpha} \left(\frac{\alpha r}{r'} \right)' \right) \right] \left(\frac{r'}{z' \alpha} \right) \frac{M_T'}{D(1-\nu)} \end{aligned} \quad (2)$$

5.3.5 (Cont'd)

where

$$L(\beta) = \frac{r'}{z'\alpha^2} \left[\beta'' + \frac{\alpha}{r} \left(\frac{r}{\alpha} \right)' \beta' + \left\{ \nu \frac{\alpha}{r} \left(\frac{r'}{\alpha} \right)' - \left(\frac{r'}{r} \right)^2 \right\} \beta \right]$$

The Equation (2) is the general differential equation governing the linear analysis for stresses, and deformations in arbitrary shells of revolution due to axisymmetric loads and temperature (see, however, remarks made in footnote, page 5.7).

(2) Force Resultant, Moment, and Displacement Boundary Conditions in Terms of the Slope β .

Since the differential equation (2) is in terms of the dependent variable β , it is advantageous to express the force resultants, moments and displacements, as well as the boundary conditions, as functions of this variable. The derived quantities are given by

$$N_\theta = \frac{D}{\alpha} \left[\frac{r}{\alpha z'} \beta'' + \frac{1}{z'} \left(\frac{r}{\alpha} \right)' \beta' + \left\{ \frac{\nu}{z'} \left(\frac{r'}{\alpha} \right)' - \frac{r'^2}{\alpha r z'} \right\} \beta \right] + \left\{ \frac{r r' V}{z'} + \frac{r M_T'}{z' (1 - \nu)} \right\}' + r P_H \quad (3a)$$

$$N_\xi = D \left[-\frac{r'}{\alpha^2 z'} \beta'' + \frac{r'}{r \alpha z'} \left(\frac{r}{\alpha} \right)' \beta' + \frac{r'}{r \alpha} \left\{ \frac{\nu}{z'} \left(\frac{r'}{\alpha} \right)' - \frac{r'^2}{\alpha r z'} \right\} \beta \right] + \frac{\alpha}{z'} V + \frac{r'}{\alpha z'} \frac{M_T'}{1 - \nu} \quad (3b)$$

$$Q = -D \left[-\frac{1}{\alpha^2} \beta'' + \frac{1}{r \alpha} \left(\frac{r}{\alpha} \right)' \beta' + \left\{ \frac{\nu}{r \alpha} \left(\frac{r'}{\alpha} \right)' - \left(\frac{r'}{\alpha r} \right)^2 \right\} \beta \right] - \frac{M_T'}{\alpha (1 - \nu)} \quad (3c)$$

$$H = \frac{D}{\alpha z'} \left[\beta'' + \frac{\alpha}{r} \left(\frac{r}{\alpha} \right)' \beta' + \left\{ \frac{\nu \alpha}{r} \left(\frac{r'}{\alpha} \right)' - \left(\frac{r'}{r} \right)^2 \right\} \beta \right] + \frac{\alpha r' V}{D} + \frac{\alpha M_T'}{D (1 - \nu)} \quad (3d)$$

$$M_\xi = -D \left[\frac{\beta'}{\alpha} + \frac{\nu r'}{r \alpha} \beta \right] - \frac{M_T}{1 - \nu} \quad (3e)$$

5.3.5 (Cont'd)

$$M_{\theta} = -D \left[\frac{r'}{r\alpha} \beta + \frac{\nu\beta'}{\alpha} \right] - \frac{M_T}{1-\nu} \quad (3f)$$

$$u = \frac{r}{Eh} (N_{\theta} - \nu N_{\xi}) + \frac{N_T r}{Eh} \quad (3g)$$

$$w = \int_{\xi_0}^{\xi} \left[\frac{z'}{Eh} \left(N_{\xi} - \nu N_{\theta} + N_T \right) + r' \beta \right] d\xi. \quad (3h)$$

The stresses may then be obtained from

$$\sigma_{\xi} = \frac{1}{h} \left(N_{\xi} + \frac{N_T}{1-\nu} \right) - \frac{E\lambda T}{1-\nu} + \frac{12\zeta}{h^3} \left(M_{\xi} + \frac{M_T}{1-\nu} \right) \quad (4a)$$

$$\sigma_{\theta} = \frac{1}{h} \left(N_{\theta} + \frac{N_T}{1-\nu} \right) - \frac{E\lambda T}{1-\nu} + \frac{12\zeta}{h^3} \left(M_{\theta} + \frac{M_T}{1-\nu} \right). \quad (4b)$$

Typical boundary conditions can be expressed in terms of the above derived quantities as shown by the following examples:

(a) Clamped Edge at $\xi = \xi_0$

$$\left[\beta \right]_{\xi = \xi_0} = \left[u \right]_{\xi = \xi_0} = 0 \quad (5a)$$

(b) Pinned Edge at $\xi = \xi_0$

$$\left[u \right]_{\xi = \xi_0} = \left[M_{\xi} \right]_{\xi = \xi_0} = 0 \quad (5b)$$

(c) Free Edge at $\xi = \xi_0$

$$\left[Q \right]_{\xi = \xi_0} = \left[M_{\xi} \right]_{\xi = \xi_0} = 0 \quad (5c)$$

(d) Specified Radial Load " H_0 " and Meridional Moment " M_0 " at edge $\xi = \xi_0$

$$\begin{aligned} & \left[\frac{D}{\alpha z'} \left(\beta'' + \frac{\alpha}{r} \left(\frac{r'}{\alpha} \right)' \beta' + \left\{ \frac{\nu\alpha}{r} \left(\frac{r'}{\alpha} \right)' - \left(\frac{r'}{r} \right)^2 \right\} \beta \right. \right. \\ & \left. \left. + \frac{\alpha r' V}{D} + \frac{\alpha M_T'}{D(1-\nu)} \right) \right]_{\xi = \xi_0} = H_0 \\ & \left[-D \left(\frac{\beta'}{\alpha} + \frac{\nu r'}{r\alpha} \beta \right) - \frac{M_T}{1-\nu} \right]_{\xi = \xi_0} = M_0 \end{aligned} \quad (5d)$$

5.3.5 (Cont'd)

In order to apply the differential equation (2) and boundary conditions (3) to a specific shell geometry, the quantities r , α , z must be determined from the equation of the shell meridian. Specific examples for the cases of conical and cylindrical shells are presented in the following paragraphs. In general, the major problem encountered in solving Eq. (2) for a given shell of revolution involves obtaining the complementary solution for a fourth order differential equation with variable coefficients. Usually closed form solutions are not possible. However, as may be observed from an inspection of the right hand side of Eq. (2), the introduction of temperature does not complicate matters, since it enters in a form analogous to terms resulting from mechanical loads.

The boundary conditions (Eq. (5d)) are of importance in considering the interaction of the shell with both transverse circular bulkheads and other shells of revolution. Once the linear response to edge loads and moments is determined from a solution of Eq. (2) subject to boundary conditions of the type (5d), influence coefficients are known and internal loads at the junctions of shells and bulkheads may be evaluated by imposing compatibility and equilibrium conditions on slopes and deflections. A description of the procedure to be followed is given in detail in Volume I, Section 8.

5.4 CONICAL SHELLS

5.4.1 Basic Equations

A meridional section of the cone and the coordinate system is shown in Figure 5.4.1-1.

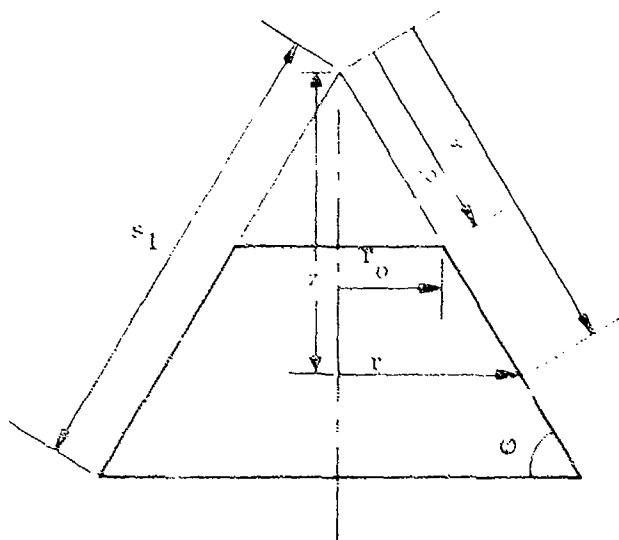


FIGURE 5.4.1-1 CONE GEOMETRY

Referring to Eqs. (1) of Paragraph 5.3.1, we choose the independent variable as

$$\xi = s \quad (13)$$

5.4.1 (Cont'd)

from which it follows that

$$\left. \begin{aligned} r &= s \cos \varphi \\ z &= s \sin \varphi \\ r' &= \cos \varphi \\ z' &= \sin \varphi \\ \alpha &= 1, \end{aligned} \right\} \quad (1b)$$

where $\varphi = \text{constant}$ has been employed and primed quantities indicate differentiation with respect to s .

The above quantities may be substituted into the general differential equation (2) of Paragraph 5.3.5. After simplification, the resulting differential equation for the linear analysis of conical shells becomes

$$\begin{aligned} s^3 IV + 4s''' + D^4 \frac{\beta}{s} + \frac{1}{D} \left(\frac{1}{s} (P_V s^2)' + \nu \tan^2 \varphi P_V' \cos \varphi \right. \\ \left. + \frac{1}{\cos \varphi} \frac{1}{s^2} \int_{s_0}^s s P_V ds + \frac{1}{s^2} (P_H s^2)' + \nu P_H' \sin \varphi \right) \\ \left. + \frac{r_0 V_0}{s^2 \cos^2 \varphi} + \tan \varphi N_T' + \frac{1}{s^2} \left(\frac{s^3 M_T''}{1-\nu} \right)' \right] = 0, \end{aligned} \quad (2)$$

where $D^4 = \frac{E h \tan^2 \varphi}{D}$.

We will consider the special case of normal surface pressure. Then, from Figure 5.3.1-1:

$$P_V = P \cos \varphi$$

$$P_H = P \sin \varphi,$$

and Eq. (2) reduces to

$$\begin{aligned} s^3 IV + 4s''' + D^4 \frac{\beta}{s} + \frac{1}{D} \left(\frac{1}{s} (P s^2)' + \frac{1}{s^2} \int_{s_0}^s s P ds + \frac{r_0 V_0}{s^2 \cos^2 \varphi} \right. \\ \left. + \tan \varphi N_T' + \frac{1}{s^2} \left(\frac{s^3 M_T''}{1-\nu} \right)' \right) = 0. \end{aligned} \quad (3)$$

5.4.1 (Cont'd)

Correspondingly, the expressions for the force resultants and moments are obtained from Eqs. (3) of Paragraph 5.3.5 and are given by

$$N_\theta = \frac{1}{\tan \varphi} \left[D \left(s\beta'' + \beta' - \frac{\beta}{s} \right)' - Ps + \frac{(sM_T)'}{1-\nu} \right] \quad (4a)$$

$$N_s = \frac{1}{\tan \varphi} \left[D \left(\beta'' + \frac{\beta'}{s} - \frac{\beta}{s^2} \right) + \frac{r_0 V_0}{\cos^2 \varphi} - \frac{1}{s} \int_{s_0}^s Ps ds + \frac{M_T}{1-\nu} \right] \quad (4b)$$

$$Q = -D \left[\beta'' + \frac{\beta'}{s} - \frac{\beta}{s^2} \right] - \frac{M_T}{1-\nu} \quad (4c)$$

$$H = \frac{D}{\sin \varphi} \left[\beta'' + \frac{\beta'}{s} - \frac{\beta}{s^2} \right] + \frac{V_0 r_0}{s \sin \varphi} + \frac{\cos^2 \varphi}{s \sin \varphi} \int_{s_0}^s Ps ds + \frac{1}{\sin \varphi} \frac{M_T}{1-\nu} \quad (4d)$$

$$M_s = -D \left[\beta' + \frac{\nu \beta}{s} \right] - \frac{M_T}{1-\nu} \quad (4e)$$

$$M_\theta = -D \left[\frac{\beta}{s} + \nu \beta' \right] - \frac{M_T}{1-\nu} \quad (4f)$$

5.4.2 Solution of the Differential Equation*

When Eq. (3) of Paragraph 5.4.1 is multiplied by s^2 , we have

$$s^2 \beta^{IV} + 4s \beta''' + 6\beta'' = H(s) + \frac{r_0 V_0}{D \sec^2 \varphi} \quad (5a)$$

where

$$H(s) = \frac{1}{D} \left[(Ps^2)' + \frac{1}{s} \int_{s_0}^s Ps ds + \frac{1}{s} \left(s^3 \frac{M_T}{1-\nu} \right)' - \tan \varphi + N_T \right] \quad (5b)$$

Introduce a new independent variable ξ defined by

$$\xi = 2\alpha s^2 \quad (5c)$$

* This differential equation for cones of revolution subjected to axisymmetric mechanical loads and temperature was solved by Bittl (Reference 5-5) where reference is given to previous work on the subject. However, the equations did not proceed from any general development on shells of revolution. Furthermore, only linear temperature variations through the thickness are accommodated in this reference.

5.4.2 (Cont'd)

Then the homogeneous form of Eq. (1) becomes

$$\frac{d^4 \beta_c}{d\psi^4} + \frac{2}{\psi} \frac{d^3 \beta_c}{d\psi^3} - \frac{9}{\psi^2} \frac{d^2 \beta_c}{d\psi^2} + \frac{9}{\psi^3} \frac{d\beta_c}{d\psi} + \beta_c = 0. \quad (3a)$$

This equation is now factorable as

$$\left(\frac{d^2}{d\psi^2} + \frac{1}{\psi} \frac{d}{d\psi} - i - \frac{4}{\psi^2} \right) \left(\frac{d^2}{d\psi^2} + \frac{1}{\psi} \frac{d}{d\psi} - i - \frac{4}{\psi^2} \right) \beta_c = 0, \quad (3b)$$

for which the solution is

$$\beta_c = c_1 J_2 \left(i^{\frac{1}{2}} \psi \right) + c_2 Y_2 \left(i^{\frac{1}{2}} \psi \right) + c_3 J_2 \left(\frac{3}{i^{\frac{1}{2}}} \psi \right) + c_4 Y_2 \left(\frac{3}{i^{\frac{1}{2}}} \psi \right), \quad (4)$$

where J_2 and Y_2 are Bessel functions of the first and second kind of order 2. After much manipulation Eq. (4) may be written in terms of Kelvin functions as

$$\begin{aligned} \beta_c = & A \left(\text{ber} \psi - \frac{2}{\psi} \dot{\text{bei}} \psi \right) + B \left(\text{bei} \psi + \frac{2}{\psi} \dot{\text{ber}} \psi \right) \\ & + C \left(\text{ker} \psi - \frac{2}{\psi} \dot{\text{kei}} \psi \right) + F \left(\text{kei} \psi + \frac{2}{\psi} \dot{\text{ker}} \psi \right) \end{aligned} \quad (5)$$

where $\dot{} = \frac{d}{d\psi}$.

The particular solution of Eq. (1) can be easily obtained if the function $H(s)$ is expressed as a polynomial of the form

$$H(s) = \sum_{p=0}^N A_p s^p \quad (6)$$

Then, selecting the solution on the form of series involving integral powers of s , there results

$$\begin{aligned} \beta_p = & \frac{r_0 V_0}{E h s \sin^2 \varphi} + \frac{A_0}{\rho^4} \\ & + \sum_{p=1}^l \sum_{m=1}^p \left[\frac{(-1)^{p-m+1}}{z} \right] a_m s^m, \end{aligned} \quad (7)$$

5.4.2 (Cont'd)

where the a_m are given by

$$a_m = (-1)^{\frac{p-m}{2}} \frac{A_p (p+1) (p) (m+1) (m)}{\rho^2 (p-m+2)} \left[\frac{(p-1)!}{(m+1)!} \right]^2 \quad (8)$$

The total solution to the cone problem is $\beta = \beta_c + \beta_p$ and the constants A, B, C, F must be determined from the four boundary conditions (two at each edge). For the special case of a full cone ($s_0 = 0$), finiteness of stresses at the apex requires that $V_0 = 0$. Further, regularity conditions on stress resultants and shears, require that "C" = "F" = 0. Specific solutions for the full cone then resolve themselves into the determination of the two constants "A" and "B" from the boundary conditions at the base.

The Kelvin functions and their first derivatives appearing in Eq. (5) are extensively and accurately tabulated in Reference 5-4 for a wide range of the argument. The form of solution developed above is readily adaptable to numerical computation using digital computers. Parametric studies to determine stresses and deflections can be made.

5.5 CYLINDRICAL SHELLS

5.5.1 Basic Equations

The cylinder and the coordinate system is shown in Figure 5.5.1-1.

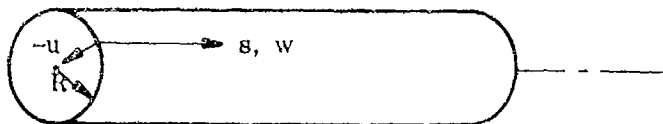


FIGURE 5.5.1-1 CYLINDRICAL GEOMETRY

We choose s again as the independent variable, however, since $r' = 0$ for the cylinder (see Figure 5.3.1-1), then as indicated previously Eq. (2) of Paragraph 5.3.5 is not applicable in the form given. The equations for the cylindrical shell are more readily derivable from the conical shell equations (which were developed from Eq. (2) of Paragraph 5.3.5). Proceeding in this manner, if the equation (3) of Paragraph 5.4.1 is multiplied by $\frac{D}{s}$, the quantity $\frac{1}{s}$ replaced by $\frac{\cos \phi}{r}$ and the appropriate limit as $\phi \rightarrow \frac{\pi}{2}$, $r \rightarrow R$, is taken, there results, for the case of normal pressures,

$$D\beta^{IV} + \frac{Eh\beta}{R^2} = \left[P - \frac{M_T''}{1-\nu} - \frac{N_T}{R} \right]' \quad (1)$$

5.5.1 (Cont'd)

Similarly, the limiting process applied to Eqs. (4) of Paragraph 5.4.1 for the force and moment resultants yields

$$\begin{aligned}
 N_{\theta} &= R \left[D\beta''' + \frac{M_T''}{1-\nu} - P \right] \\
 N_s &= V_0 \\
 Q &= - \left[D\beta'' + \frac{M_T'}{1-\nu} \right] = -H \\
 M_s &= -D\beta' - \frac{M_T}{1-\nu} \\
 M_{\theta} &= -\nu D\beta' - \frac{M_T}{1-\nu} .
 \end{aligned} \tag{2}$$

An alternate form of the equilibrium equation in terms of the radial displacement component "u" can be obtained by integrating Eq. (1) once, noting, that $\beta = -u'$. This results in

$$Du^{IV} + \frac{Eh}{R^2} u = -P + \frac{M_T''}{1-\nu} + \frac{N_T}{R} - \frac{\nu V_0}{R} , \tag{3}$$

where the constant of integration is given by the last term on the right. Substituting $\beta = -u'$ into Eqs. (2) the force and moment resultants are

$$\begin{aligned}
 N_{\theta} &= R \left[-Du^{IV} + \frac{M_T''}{1-\nu} - P \right] \\
 N_s &= V_0 \\
 Q &= Du''' - \frac{M_T'}{1-\nu} = -H \\
 M_s &= Du'' - \frac{M_T}{1-\nu} \\
 M_{\theta} &= \nu Du' - \frac{M_T}{1-\nu} .
 \end{aligned} \tag{4}$$

Using the formulation of Eq. (1), for all sets of boundary conditions involving β , the solution may be integrated once to obtain u. The constant of integration is determined from

5.5.1 (Cont'd)

the prescribed value of V_o . When using the formulation of Eq. (3), however, V_o appears explicitly in the differential equation.

5.5.2 Solution of the Differential Equation

Equation (3) may be written as

$$\frac{t^4}{4} u^{IV} + u = G(s) \quad (1a)$$

where

$$t = \frac{\sqrt{Rh}}{\sqrt{3(1-\nu)^2}} \quad (= .778 \sqrt{Rh} \text{ for } \nu = .30).$$

and

$$G(s) = \frac{R^2}{Eh} \left[-P + \frac{M_T''}{1-\nu} + \frac{N_T}{R} - \frac{\nu V_o}{R} \right]. \quad (1b)$$

The complementary solution of (1a) is given by

$$u_c = \sinh \frac{s}{t} \left(A_1 \sin \frac{s}{t} + A_2 \cos \frac{s}{t} \right) + \cosh \frac{s}{t} \left(A_3 \sin \frac{s}{t} + A_4 \cos \frac{s}{t} \right). \quad (2)$$

A particular solution can easily be obtained if $G(s)$ is expressed as a polynomial with terms of the form

$$G_k(s) = C_k \left(\frac{s}{R} \right)^k; \quad k = 0, 1, 2 \dots M. \quad (3)$$

Then a particular solution corresponding to $G(s) = C_k \left(\frac{s}{R} \right)^k$ can be determined by assuming this solution in the form of a polynomial in (s/R) of degree K and substituting in Eq. (1). There results

$$(u_p)_k = \begin{cases} C_k \left(\frac{s}{R} \right)^k & ; k \leq 3 \\ [N] \sum_{j=0}^{k-4} A_{(k-4j)} \left(\frac{s}{R} \right)^{(k-4j)} & ; k > 3, \end{cases} \quad (4)$$

5.5.2 (Cont'd)

where

$$A_{(k-4j)} = (-1)^j C_k \frac{(k)!}{(k-4j)!} \left[\frac{1}{4} \left(\frac{l}{R} \right)^4 \right]^j,$$

and

$$[N] = \text{Greatest integer} \leq \frac{k}{4}.$$

The complete solution corresponding to $G(s) = \sum_{k=0}^M C_k \left(\frac{s}{R} \right)^k$ is then given by

$$u = u_c + \sum_{k=0}^M (u_p)_k. \quad (5)$$

The constants A_1, A_2, A_3, A_4 must be determined from the specified boundary conditions at the ends of the cylinder. This requires the solution of four simultaneous, linear algebraic equations. In order to eliminate the necessity of solving four equations an approximate method of solution in which the less tedious procedure of solving two pairs of simultaneous equations, each for two unknown constants, is used extensively. The basis of this method is the assumption that edge shears and moments (which are self equilibrating) applied at one end of the shell negligibly affect the stresses and deflections at the opposite end. This assumption is valid if the length of the shell "L" satisfies the condition $L > 3l$, or equivalently, for $\nu = 0.30$, when

$$\frac{L}{R} \geq \frac{7}{3} \left(\frac{h}{R} \right)^{1/2}. \quad \text{The inequality is satisfied in all but very short/thick cylindrical shells.}$$

The method, which appears in Reference 5-2, may be described as follows:

(1) Determine the deflections and stresses corresponding to the particular solution $u_p = \sum_{k=0}^M (u_p)_k$ where the $(u_p)_k$ are given by Eq. (4). In general this solution will not satisfy the boundary conditions.

(2) Edge moments and loads are then applied to the semi-infinite cylinder such that when their effects are superposed on the solutions of (1) the boundary conditions at each edge are satisfied. Since there are only two conditions to be met at each edge and there is no assumed interaction of edge effects then, for each edge, the solution of two simultaneous equations in two unknowns (one shear and one bending moment) is required.

5.5.3 NUMERICAL EXAMPLE

A free cylindrical shell is subjected to an elevated temperature as shown in Figure 5.5.3-1, i.e., the inside surface of the cylinder is at a uniform elevated temperature T_i and the outside temperature varies linearly from T_i at $s=0$ to a temperature $T_i(1+A)$ at $s=L$. The

5.5.3 (Cont'd)

variation through the thickness of the shell is assumed to be linear. It is required to determine the stresses and deflections.

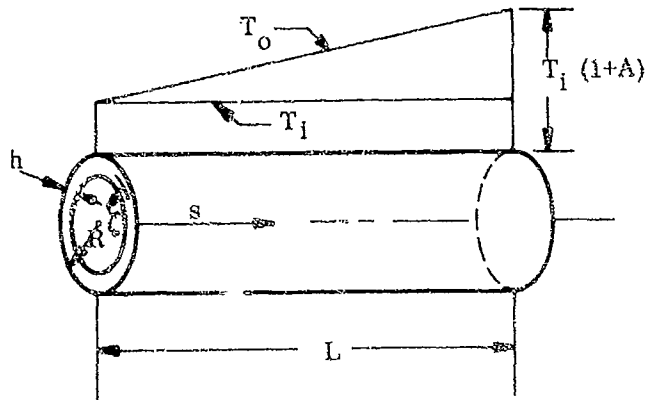


FIGURE 5.5.3-1 ILLUSTRATIVE EXAMPLE

Procedure:

The temperature distribution is given by

$$T(s, z) = T_i \left[1 + \frac{As}{2L} \left(1 - \frac{2z}{h} \right) \right]$$

and therefore

$$N_T = E\lambda \int_{-\frac{h}{2}}^{\frac{h}{2}} T dz = E\lambda T_i h \left(1 + \frac{As}{2L} \right)$$

$$M_T = E\lambda \int_{-\frac{h}{2}}^{\frac{h}{2}} T z dz = -E\lambda T_i \frac{As}{L} \frac{h^2}{12}$$

Referring to Eq. (1b) of Paragraph 5.5.2, since no mechanical loads are applied, $P = V_0 = 0$, and Eq. (1a) of Paragraph 5.5.2 becomes $\frac{1}{4} u^{IV} + u = \frac{R^2}{Eh} \left[\frac{M_T''}{1-\nu} + \frac{N_T}{R} \right]$
 $= R\lambda T_i \left(1 + \frac{As}{2L} \right).$

The right hand side can be written in the form $C_0 + C_1 \frac{s}{R}$,

where

$$C_0 = R\lambda T_i$$

$$C_1 = \frac{R^2 \lambda T_i A}{2L}.$$

5.5.3 (Cont'd)

Then from Eqs. (2), (4), and (5) of Paragraph 5.5.2 the solution for the displacement is given by

$$u = \sinh \frac{s}{l} \left(A_1 \sin \frac{s}{l} + A_2 \cos \frac{s}{l} \right) + \cosh \frac{s}{l} \left(A_3 \sin \frac{s}{l} + A_4 \cos \frac{s}{l} \right) + R\lambda T_i \left(1 + \frac{As}{2L} \right) \quad (1)$$

The boundary conditions for free edges are

$$M_s = Q = 0 \quad \text{at } s = 0, L,$$

which from Eqs. 4 of Paragraph 5.5.1 can be written as

$$Du'' = \frac{M_T}{1-\nu} = Du''' = \frac{M_T}{1-\nu} = 0 \quad \text{at } s = 0, L. \quad (2)$$

The constants $A_1 - A_4$ in Eq. (1) are determined from the boundary conditions.

Force resultants and moments are then found by substituting into Eqs. (2) of Paragraph 5.5.1. Since the calculation details are straightforward, only the results are shown. Figure 5.5.3-2 gives nondimensional deflections, force and moment resultants for both a cylinder with $L/l = 10$ and a longer cylinder corresponding to $L/l = 50$. The graphs show that as the cylinder becomes longer, the peak deflections and stresses approach the end of the surface subjected to the higher thermal gradient. In general, these peak values tend to increase in magnitude for the longer cylinders, resulting in sharp gradients in the vicinity of the edge.

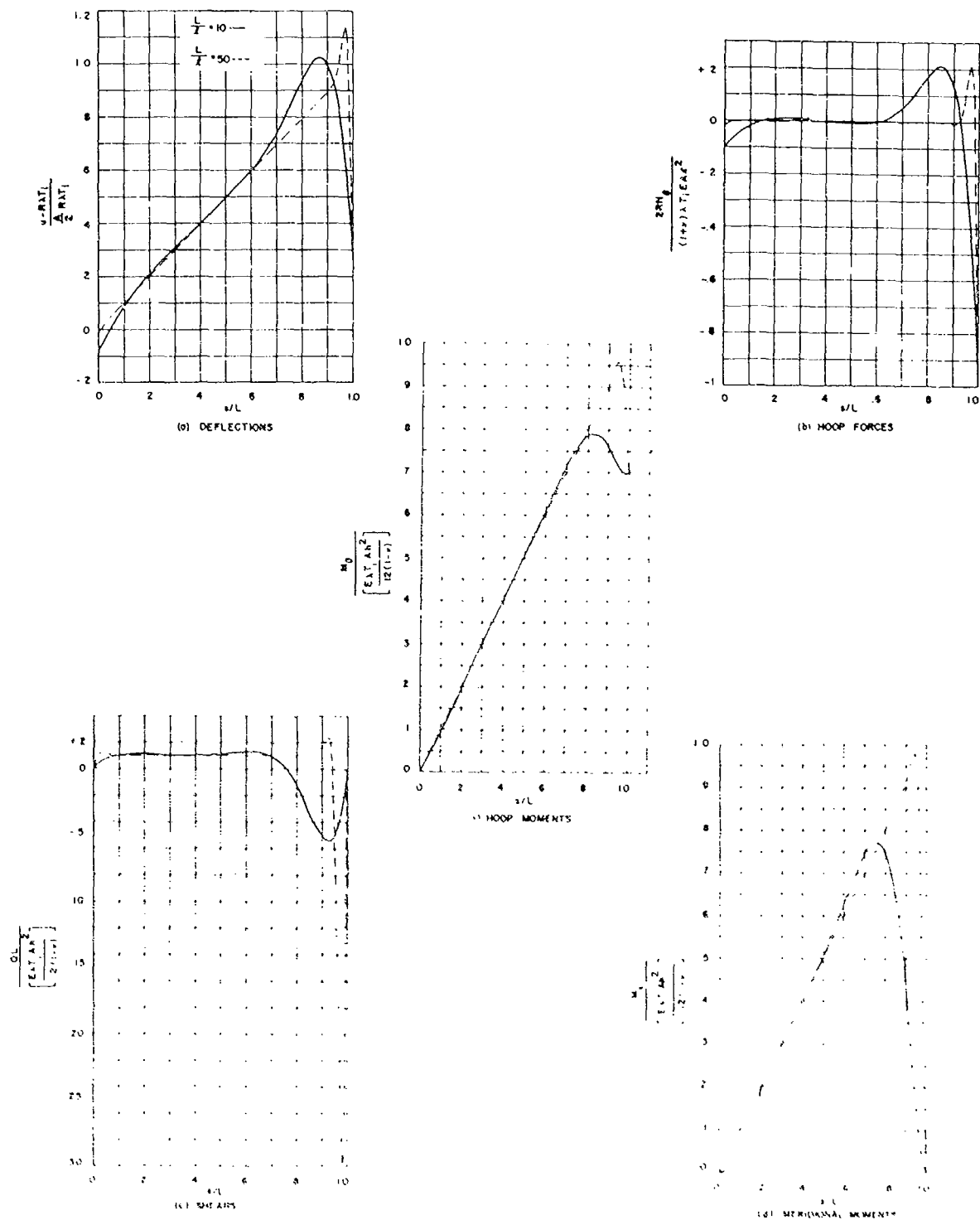


FIGURE 5.5.3-2 NONDIMENSIONAL DEFLECTIONS, FORCE AND MOMENT RESULTANTS FOR THE FREE CYLINDER HEATED AS SHOWN IN FIGURE 5.5.3-1: $l/t = 10, 50$.

5.6 REFERENCES

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- 5-2 Switzky, H., Forray, M., and Newman, M., "Thermo-Structural Analysis Manual"-Volume I, Republic Aviation Corporation Report No. 679-1, September 1960, revised November 1961 (to be published as WADD TR 60-517, Vol. I).
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SECTION 1 - BEAM COLUMNS

Beam columns with axial and restraints and end loads coupled with transverse load and temperature gradients through the depth are analyzed.

SECTIONS 2 and 3 - ECCENTRIC COLUMNS

Nondimensional curves are presented for the approximate analysis of the buckling and load

determination of restrained columns with eccentricities induced by mechanical loads and thermal environment.

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Large deflections of axially restrained, heated and loaded circular plates are treated.

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The general equation for linear thermo-elastic analysis of axisymmetric shells is developed. Cylinders and cones are discussed.

1. Thermomechanical analysis
2. Structures
3. Structural shells
4. Thermal stresses
5. Mechanical stresses
I. AFSC Project 1367, Task 136710
II. Contract
AF33(616)-6066

III. Republic Aviation Corp., R & D Div., Farmingdale, N.Y.
IV. M.J. Forray, M. Newman and H. Switzky

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VI. Avail for OTS

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